

# Geometric theta-lifting for the dual pair $\mathrm{GSp}_{2n}, \mathrm{GO}_{2m}$

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**ABSTRACT** Let  $X$  be a smooth projective curve over an algebraically closed field of characteristic  $> 2$ . Consider the dual pair  $H = \mathrm{GO}_{2m}, G = \mathrm{GSp}_{2n}$  over  $X$ , where  $H$  splits over an étale two-sheeted covering  $\pi : \tilde{X} \rightarrow X$ . Write  $\mathrm{Bun}_G$  and  $\mathrm{Bun}_H$  for the stacks of  $G$ -torsors and  $H$ -torsors on  $X$ . We show that for  $m \leq n$  (respectively, for  $m > n$ ) the theta-lifting functor  $F_G : \mathrm{D}(\mathrm{Bun}_H) \rightarrow \mathrm{D}(\mathrm{Bun}_G)$  (respectively,  $F_H : \mathrm{D}(\mathrm{Bun}_G) \rightarrow \mathrm{D}(\mathrm{Bun}_H)$ ) commutes with Hecke functors with respect to a morphism of the corresponding L-groups involving the  $\mathrm{SL}_2$  of Arthur. In two particular cases  $n = m$  and  $m = n + 1$  this becomes the geometric Langlands functoriality for the corresponding dual pair.

As an application, we prove a particular case of the geometric Langlands conjectures. Namely, we construct the automorphic Hecke eigensheaves on  $\mathrm{Bun}_{\mathrm{GSp}_4}$  corresponding to the endoscopic local systems on  $X$ .

## 1. INTRODUCTION

1.1 The classical theta correspondence for the dual reductive pair  $(\mathrm{GSp}_{2n}, \mathrm{GO}_{2m})$  is known to satisfy a version of strong Howe duality (cf. [12]). In this paper, which is a continuation of [7], we develop the geometric theory of theta-lifting for this dual pair in the everywhere unramified case.

The classical theta-lifting operators for this dual pair are as follows. Let  $X$  be a smooth projective geometrically connected curve over  $\mathbb{F}_q$  (with  $q$  odd). Let  $F = \mathbb{F}_q(X)$ ,  $A$  be the adèles ring of  $X$ ,  $\mathcal{O}$  the integer adèles. Write  $\Omega$  for the canonical line bundle on  $X$ . Pick a rank  $2n$ -vector bundle  $M$  with symplectic form  $\wedge^2 M \rightarrow \mathcal{A}$  with values in a line bundle  $\mathcal{A}$  on  $X$ . Let  $G$  be the group scheme over  $X$  of automorphisms of the  $\mathrm{GSp}_{2n}$ -torsor  $(M, \mathcal{A})$ .

Let  $\pi : \tilde{X} \rightarrow X$  be an étale two-sheeted covering with Galois group  $\Sigma = \{1, \sigma\}$ . Let  $\mathcal{E}$  be the  $\sigma$ -anti-invariants in  $\pi_* \mathcal{O}_{\tilde{X}}$ . Fix a rank  $2m$ -vector bundle  $V$  on  $X$  with symmetric form  $\mathrm{Sym}^2 V \rightarrow \mathcal{C}$  with values in a line bundle  $\mathcal{C}$  on  $X$  together with a compatible trivialization  $\gamma : \mathcal{C}^{-m} \otimes \det V \xrightarrow{\sim} \mathcal{E}$ . This means that  $\gamma^2 : \mathcal{C}^{-2m} \otimes (\det V)^2 \xrightarrow{\sim} \mathcal{O}$  is the trivialization induced by the symmetric form. Let  $\tilde{H}$  be the group scheme over  $X$  of automorphisms of  $V$  preserving the symmetric form up to a multiple and fixing  $\gamma$ . This is a form of  $\mathrm{GO}_{2m}^0$ , where  $\mathrm{GO}_{2m}^0$  is the connected component of unity of the split orthogonal similitude group. Assume given an isomorphism  $\mathcal{A} \otimes \mathcal{C} \xrightarrow{\sim} \Omega$ .

Let  $G_{2nm}$  the group scheme of automorphisms of  $M \otimes V$  preserving the symplectic form  $\wedge^2(M \otimes V) \rightarrow \Omega$ . Write  $G\tilde{H} \subset G \times \tilde{H}$  for the group subscheme over  $X$  of pairs  $(g, h)$  such that  $g \otimes h$  acts trivially on  $\mathcal{A} \otimes \mathcal{C}$ . The metaplectic cover  $\tilde{G}_{2nm}(\mathbb{A}) \rightarrow G_{2nm}(\mathbb{A})$  splits naturally after restriction under  $G\tilde{H}(\mathbb{A}) \rightarrow G_{2nm}(\mathbb{A})$ . Let  $S$  be the corresponding Weil representation

of  $G\tilde{H}(\mathbb{A})$ . The space  $S^{G\tilde{H}(\mathcal{O})}$  has a distinguished nonramified vector  $v_0$ . If  $\theta : S \rightarrow \bar{\mathbb{Q}}_\ell$  is a theta-functional then  $\phi_0 : G\tilde{H}(F) \backslash G\tilde{H}(\mathbb{A}) / G\tilde{H}(\mathcal{O}) \rightarrow \bar{\mathbb{Q}}_\ell$  given by  $\phi_0(g, h) = \theta((g, h)v_0)$  is the classical theta-function. The theta-lifting operators

$$F_G : \text{Funct}(\tilde{H}(F) \backslash \tilde{H}(\mathbb{A}) / \tilde{H}(\mathcal{O})) \rightarrow \text{Funct}(G(F) \backslash G(\mathbb{A}) / G(\mathcal{O}))$$

and

$$F_{\tilde{H}} : \text{Funct}(G(F) \backslash G(\mathbb{A}) / G(\mathcal{O})) \rightarrow \text{Funct}(\tilde{H}(F) \backslash \tilde{H}(\mathbb{A}) / \tilde{H}(\mathcal{O}))$$

are the integral operators with kernel  $\phi_0$  for the diagram of projections

$$\begin{array}{ccc} & G\tilde{H}(F) \backslash G\tilde{H}(\mathbb{A}) / G\tilde{H}(\mathcal{O}) & \\ \swarrow \mathfrak{q} & & \searrow \mathfrak{p} \\ \tilde{H}(F) \backslash \tilde{H}(\mathbb{A}) / \tilde{H}(\mathcal{O}) & & G(F) \backslash G(\mathbb{A}) / G(\mathcal{O}) \end{array}$$

The following statement would be an analog of a theorem of Rallis [11] for similitude groups (the author have not found its proof in the litterature). If  $m \leq n$  (resp.,  $m > n$ ) then  $F_G$  (resp.,  $F_{\tilde{H}}$ ) commutes with the actions of global Hecke algebras  $\mathcal{H}_G, \mathcal{H}_{\tilde{H}}$  with respect to certain homomorphism  $\mathcal{H}_G \rightarrow \mathcal{H}_{\tilde{H}}$  (resp.,  $\mathcal{H}_{\tilde{H}} \rightarrow \mathcal{H}_G$ ). We prove a geometric version of this result (cf. Theorem 1). Its precise formulation in the geometric setting involves the  $\text{SL}_2$  of Arthur (or rather its maximal torus). In the particular case  $n = m$  (resp.,  $m = n + 1$ ) the  $\text{SL}_2$  of Arthur dissapears, and the corresponding morphisms of Hecke algebras come from morphisms of L-groups  $H^L \rightarrow G^L$  (resp.,  $G^L \rightarrow H^L$ ).

Our methods extend those of [7], the global results are derived from the corresponding local ones. Remind that  $S \widetilde{\otimes}'_{x \in X} S_x$  is the restricted tensor product of local Weil representations. Let  $F_x$  be the completion of  $F$  at  $x \in X$ ,  $\mathcal{O}_x \subset F_x$  the ring of integers. The geometric analog of the  $G\tilde{H}(F_x)$ -representation  $S_x$  is the Weil category  $W(\mathcal{L}_d(W_0(F_x)))$  (cf. Sections 3.1-3.2). Informally speaking, we work rather with the geometric analog of the compactly induced representation

$$\bar{S}_x = \text{c-ind}_{G\tilde{H}(F_x)}^{(G \times \tilde{H})(F_x)} S_x$$

Its manifestation is a family of categories  $\text{D}_{\mathcal{T}_a}(\tilde{\mathcal{L}}_d(W_a(F_x)))$  indexed by  $a \in \mathbb{Z}$  (cf. Section 4.2).

Our main local result is Theorem 3. In classical terms, it compares the action of Hecke operators for  $G$  and  $\tilde{H}$  on the natural nonramified vector in  $\bar{S}_x$ . As a byproduct, we also obtain some new results at the classical level of functions (Propositions A.1 and A.2). For  $a$  even they reduce to a result from [10], but for  $a$  odd they are new and amount to a calculation of  $K \times \text{SO}(\mathcal{O}_x)$ -invariants in the Weil representation of  $(\text{Sp}_{2n} \times \text{SO}_{2m})(F_x)$ , where  $K$  is the nonstandard maximal compact subgroup of  $\text{Sp}_{2n}(F_x)$ .

1.2 The most striking application of our Theorem 1 is a proof of the following particular case of the geometric Langlands conjecture for  $G = \text{GSp}_4$ . Let  $E$  be an irreducible rank 2 smooth  $\bar{\mathbb{Q}}_\ell$ -sheaf on  $\tilde{X}$  equipped with an isomorphism  $\pi^* \chi \xrightarrow{\sim} \det E$ , where  $\chi$  is a smooth  $\bar{\mathbb{Q}}_\ell$ -sheaf on  $X$  of rank one. Then  $\pi_*(E^*)$  is equipped with a natural symplectic form  $\wedge^2(\pi_* E^*) \rightarrow \chi^{-1}$ , so can be viewed as a  $\check{G}$ -local system  $E_{\check{G}}$  on  $X$ , where  $\check{G}$  is the Langlands dual group over  $\bar{\mathbb{Q}}_\ell$ . We

construct the automorphic sheaf  $K$  on  $\text{Bun}_G$ , which is a Hecke eigensheaf with respect to  $E_{\check{G}}$  (cf. Corollary 1).

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## 2. MAIN RESULTS

**2.1 NOTATION** From now on  $k$  denotes an algebraically closed field of characteristic  $p > 2$ , all the schemes (or stacks) we consider are defined over  $k$  (except in Section 4.8.7.2).

Fix a prime  $\ell \neq p$ . For a scheme (or stack)  $S$  write  $\text{D}(S)$  for the bounded derived category of  $\ell$ -adic étale sheaves on  $S$ , and  $\text{P}(S) \subset \text{D}(S)$  for the category of perverse sheaves. Set  $\text{DP}(S) = \bigoplus_{i \in \mathbb{Z}} \text{P}(S)[i] \subset \text{D}(S)$ . By definition, we let for  $K, K' \in \text{P}(S), i, j \in \mathbb{Z}$

$$\text{Hom}_{\text{DP}(S)}(K[i], K'[j]) = \begin{cases} \text{Hom}_{\text{P}(S)}(K, K'), & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

Since we are working over an algebraically closed field, we systematically ignore Tate twists (except in Section 4.8.7.2, where we work over a finite subfield  $k_0 \subset k$ . In this case we also fix a square root  $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$  of the sheaf  $\bar{\mathbb{Q}}_\ell(1)$  over  $\text{Spec } k_0$ ). Fix a nontrivial character  $\psi : \mathbb{F}_p \rightarrow \bar{\mathbb{Q}}_\ell^*$  and denote by  $\mathcal{L}_\psi$  the corresponding Artin-Shreier sheaf on  $\mathbb{A}^1$ .

If  $V \rightarrow S$  and  $V^* \rightarrow S$  are dual rank  $n$  vector bundles over a stack  $S$ , we normalize the Fourier transform  $\text{Four}_\psi : \text{D}(V) \rightarrow \text{D}(V^*)$  by  $\text{Four}_\psi(K) = (p_{V^*})_!(\xi^* \mathcal{L}_\psi \otimes p_V^* K)[n](\frac{n}{2})$ , where  $p_V, p_{V^*}$  are the projections, and  $\xi : V \times_S V^* \rightarrow \mathbb{A}^1$  is the pairing.

For a sheaf of groups  $G$  on a scheme  $S$ ,  $\mathcal{F}_G^0$  denotes the trivial  $G$ -torsor on  $S$ . For a representation  $V$  of  $G$  and a  $G$ -torsor  $\mathcal{F}_G$  on  $S$  write  $V_{\mathcal{F}_G} = V \times^G \mathcal{F}_G$  for the induced vector bundle on  $S$ . For a morphism of stacks  $f : Y \rightarrow Z$  denote by  $\dim.\text{rel}(f)$  the function of connected component  $C$  of  $Y$  given by  $\dim C - \dim C'$ , where  $C'$  is the connected component of  $Z$  containing  $f(C)$ .

**2.2 HECKE OPERATORS** Let  $X$  be a smooth connected projective curve. For  $r \geq 1$  write  $\text{Bun}_r$  for the stack of rank  $r$  vector bundles on  $X$ . The Picard stack  $\text{Bun}_1$  is also denoted  $\text{Pic } X$ . For a connected reductive group  $\mathbb{G}$  over  $k$ , let  $\text{Bun}_{\mathbb{G}}$  denote the stack of  $\mathbb{G}$ -torsors on  $X$ .

Given a maximal torus and a Borel subgroup  $\mathbb{T} \subset \mathbb{B} \subset \mathbb{G}$ , we write  $\Lambda_{\mathbb{G}}$  (resp.,  $\check{\Lambda}_{\mathbb{G}}$ ) for the coweights (resp., weights) lattice of  $\mathbb{G}$ . Let  $\Lambda_{\mathbb{G}}^+$  (resp.,  $\check{\Lambda}_{\mathbb{G}}^+$ ) denote the set of dominant coweights (resp., dominant weights) of  $\mathbb{G}$ . Write  $\check{\rho}_{\mathbb{G}}$  (resp.,  $\rho_{\mathbb{G}}$ ) for the half sum of the positive roots (resp., coroots) of  $\mathbb{G}$ ,  $w_0$  for the longest element of the Weyl group of  $\mathbb{G}$ .

Set  $K = k(X)$ . For a closed point  $x \in X$  let  $K_x$  be the completion of  $K$  at  $x$ ,  $\mathcal{O}_x \subset K_x$  be its ring of integers.

The following notations are borrowed from [7]. Write  $\text{Gr}_{\mathbb{G}, x}$  for the affine grassmanian  $\mathbb{G}(K_x)/\mathbb{G}(\mathcal{O}_x)$ . This is an ind-scheme classifying a  $\mathbb{G}$ -torsor  $\mathcal{F}_{\mathbb{G}}$  on  $X$  together with a trivialization  $\beta : \mathcal{F}_{\mathbb{G}}|_{X-x} \xrightarrow{\sim} \mathcal{F}_{\mathbb{G}}^0|_{X-x}$ . For  $\lambda \in \Lambda_{\mathbb{G}}^+$  write  $\overline{\text{Gr}}_{\mathbb{G}, x}^\lambda \subset \text{Gr}_{\mathbb{G}, x}$  for the closed subscheme classifying  $(\mathcal{F}_{\mathbb{G}}, \beta)$  for which  $V_{\mathcal{F}_{\mathbb{G}}^0}(-\langle \lambda, \check{\lambda} \rangle x) \subset V_{\mathcal{F}_{\mathbb{G}}}$  for every  $\mathbb{G}$ -module  $V$  whose weights are  $\leq \check{\lambda}$ . The unique dense open  $\mathbb{G}(\mathcal{O}_x)$ -orbit in  $\overline{\text{Gr}}_{\mathbb{G}, x}^\lambda$  is denoted  $\text{Gr}_{\mathbb{G}, x}^\lambda$ .

For  $\theta \in \pi_1(\mathbb{G})$  the connected component  $\mathrm{Gr}_{\mathbb{G}}^{\theta}$  of  $\mathrm{Gr}_{\mathbb{G}}$  classifies pairs  $(\mathcal{F}_{\mathbb{G}}, \beta)$  such that  $V_{\mathcal{F}_{\mathbb{G}}}^0(-\langle \theta, \check{\lambda} \rangle) \xrightarrow{\sim} V_{\mathcal{F}_{\mathbb{G}}}$  for every one-dimensional  $\mathbb{G}$ -module with highest weight  $\check{\lambda}$ .

Denote by  $\mathcal{A}_{\mathbb{G}}^{\lambda}$  the intersection cohomology sheaf of  $\overline{\mathrm{Gr}}_{\mathbb{G}}^{\lambda}$ . Write  $\check{\mathbb{G}}$  for the Langlands dual group to  $\mathbb{G}$ , this is a reductive group over  $\bar{\mathbb{Q}}_{\ell}$  equipped with the dual maximal torus and Borel subgroup  $\check{\mathbb{T}} \subset \check{\mathbb{B}} \subset \check{\mathbb{G}}$ . Write  $\mathrm{Sph}_{\mathbb{G}}$  for the category of  $\mathbb{G}(\mathcal{O}_x)$ -equivariant perverse sheaves on  $\mathrm{Gr}_{\mathbb{G},x}$ . This is a tensor category, and one has a canonical equivalence of tensor categories  $\mathrm{Loc} : \mathrm{Rep}(\check{\mathbb{G}}) \xrightarrow{\sim} \mathrm{Sph}_{\mathbb{G}}$ , where  $\mathrm{Rep}(\check{\mathbb{G}})$  is the category of finite-dimensional representations of  $\check{\mathbb{G}}$  over  $\bar{\mathbb{Q}}_{\ell}$  (cf. [9]).

For the definition of the Hecke functors

$$H_{\mathbb{G}}^{\leftarrow}, H_{\mathbb{G}}^{\rightarrow} : \mathrm{Sph}_{\mathbb{G}} \times \mathrm{D}(\mathrm{Bun}_{\mathbb{G}}) \rightarrow \mathrm{D}(X \times \mathrm{Bun}_{\mathbb{G}})$$

we refer the reader to ([7], Section 2.2.1). Write  $* : \mathrm{Sph}_{\mathbb{G}} \xrightarrow{\sim} \mathrm{Sph}_{\mathbb{G}}$  for the covariant equivalence induced by the map  $\mathbb{G}(K_x) \rightarrow \mathbb{G}(K_x)$ ,  $g \mapsto g^{-1}$ . In view of  $\mathrm{Loc}$ , the corresponding functor  $* : \mathrm{Rep}(\check{\mathbb{G}}) \xrightarrow{\sim} \mathrm{Rep}(\check{\mathbb{G}})$  sends an irreducible  $\check{\mathbb{G}}$ -module with h.w.  $\lambda$  to the irreducible  $\check{\mathbb{G}}$ -module with h.w.  $-w_0(\lambda)$ . For  $\lambda \in \Lambda_{\mathbb{G}}^+$  we also write  $H_{\mathbb{G}}^{\lambda}(\cdot) = H_{\mathbb{G}}^{\leftarrow}(\mathcal{A}_{\mathbb{G}}^{\lambda}, \cdot)$ .

Set

$$\mathrm{D Sph}_{\mathbb{G}} = \bigoplus_{r \in \mathbb{Z}} \mathrm{Sph}_{\mathbb{G}}[r] \subset \mathrm{D}(\mathrm{Gr}_{\mathbb{G}})$$

As in ([7], Section 2.2.2), we equip it with a structure of a tensor category in such a way that the Satake equivalence extends to an equivalence of tensor categories  $\mathrm{Loc}^{\tau} : \mathrm{Rep}(\check{\mathbb{G}} \times \mathbb{G}_m) \xrightarrow{\sim} \mathrm{D Sph}_{\mathbb{G}}$ . Our convention is that  $\mathbb{G}_m$  acts on  $\mathrm{Sph}_{\mathbb{G}}[r]$  by the character  $x \mapsto x^{-r}$ .

Now let  $\pi : \tilde{X} \rightarrow X$  be a finite étale Galois covering with Galois group  $\Sigma$ . Given a homomorphism  $\Sigma \rightarrow \mathrm{Aut}(\mathbb{G})$ , let  $G$  be the group scheme on  $X$  obtained as the twisting of  $\mathbb{G}$  by the  $\Sigma$ -torsor  $\pi : \tilde{X} \rightarrow X$ . Set  $\tilde{K} = k(\tilde{X})$ . For a closed point  $\tilde{x} \in \tilde{X}$  write  $K_{\tilde{x}}$  for the completion of  $\tilde{K}$  at  $\tilde{x}$ ,  $\mathcal{O}_{\tilde{x}} \subset K_{\tilde{x}}$  for its ring of integers, and  $\mathrm{Gr}_{G,\tilde{x}}$  for the affine grassmanian  $\mathbb{G}(K_{\tilde{x}})/\mathbb{G}(\mathcal{O}_{\tilde{x}})$ .

Write  $\mathrm{Bun}_G$  for the stack of  $G$ -torsors on  $X$ . One defines Hecke functors

$${}_{\tilde{x}}H_G^{\leftarrow}, {}_{\tilde{x}}H_G^{\rightarrow} : \mathrm{Sph}_{\mathbb{G}} \times \mathrm{D}(\mathrm{Bun}_G) \rightarrow \mathrm{D}(\mathrm{Bun}_G) \quad (1)$$

as follows. Write  ${}_{\tilde{x}}\mathcal{H}_G$  for the Hecke stack classifying  $G$ -torsors  $\mathcal{F}_G, \mathcal{F}'_G$  on  $X$  and an isomorphism  $\mathcal{F}_G \xrightarrow{\sim} \mathcal{F}'_G|_{X-\pi(\tilde{x})}$ . We have a diagram

$$\mathrm{Bun}_G \xleftarrow{h^{\leftarrow}} {}_{\tilde{x}}\mathcal{H}_G \xrightarrow{h^{\rightarrow}} \mathrm{Bun}_G,$$

where  $h^{\leftarrow}$  (resp.,  $h^{\rightarrow}$ ) sends  $(\mathcal{F}_G, \mathcal{F}'_G, \tilde{x})$  to  $\mathcal{F}_G$  (resp., to  $\mathcal{F}'_G$ ). Set  $D_{\tilde{x}} = \mathrm{Spec} \mathcal{O}_{\tilde{x}}$ . Let  $\mathrm{Bun}_{G,\tilde{x}}$  be the stack classifying  $\mathcal{F}_G \in \mathrm{Bun}_G$  together with a trivialization  $\mathcal{F}_G|_{D_{\tilde{x}}} \xrightarrow{\sim} \mathcal{F}_{\mathbb{G}}^0$ . Write  $\mathrm{id}^l, \mathrm{id}^r$  for the isomorphisms

$${}_{\tilde{x}}\mathcal{H}_G \xrightarrow{\sim} \mathrm{Bun}_{G,\tilde{x}} \times^{\mathbb{G}(\mathcal{O}_{\tilde{x}})} \mathrm{Gr}_{G,\tilde{x}}$$

such that the projection to the first factor corresponds to  $h^{\leftarrow}, h^{\rightarrow}$  respectively. To  $\mathcal{S} \in \mathrm{Sph}_{\mathbb{G}}$ ,  $K \in \mathrm{D}(\mathrm{Bun}_G)$  one attaches their twisted external product  $(K \boxtimes \mathcal{S})^l$  and  $(K \boxtimes \mathcal{S})^r$  on  ${}_{\tilde{x}}\mathcal{H}_G$ , they are normalized to be perverse for  $K, \mathcal{S}$  perverse. The functors (1) are defined by

$${}_{\tilde{x}}H_G^{\leftarrow}(\mathcal{S}, K) = h_!^{\leftarrow}(K \boxtimes * \mathcal{S})^r \quad \text{and} \quad {}_{\tilde{x}}H_G^{\rightarrow}(\mathcal{S}, K) = h_!^{\rightarrow}(K \boxtimes \mathcal{S})^l$$

We have canonically  ${}_{\tilde{x}}\mathrm{H}_G^-(\ast\mathcal{S}, K) \xrightarrow{\sim} {}_{\tilde{x}}\mathrm{H}_G^-(\mathcal{S}, K)$ . Letting  $\tilde{x}$  move along  $\tilde{X}$ , one similarly defines Hecke functors

$$\mathrm{H}_G^-, \mathrm{H}_G^+ : \mathrm{Sph}_{\mathbb{G}} \times \mathrm{D}(\mathrm{Bun}_G) \rightarrow \mathrm{D}(\tilde{X} \times \mathrm{Bun}_G)$$

They are compatible with the tensor structure on  $\mathrm{Sph}_{\mathbb{G}}$  and commute with the Verdier duality (cf. [3, 7]). The group  $\Sigma$  acts on  $\mathrm{Gr}_{G, \tilde{x}}$ , hence also on  $\mathrm{Sph}_{\mathbb{G}}$  by transport of structure, and for  $\sigma \in \Sigma$  we have isomorphisms of functors  $(\sigma \times \mathrm{id})^* \circ \mathrm{H}_G^-(\mathcal{S}, \cdot) \xrightarrow{\sim} \mathrm{H}_G^-(\sigma^*\mathcal{S}, \cdot)$ .

Assume that  $\mathbb{T}$  is  $\Sigma$ -invariant then  $\Sigma$  acts on the root datum  $\mathcal{R} = (\Lambda_{\mathbb{G}}, R, \check{\Lambda}_{\mathbb{G}}, \check{R})$  of  $(\mathbb{G}, \mathbb{T})$ , here  $R$  and  $\check{R}$  stand for coroots and roots of  $\mathbb{G}$  respectively. Given an action of  $\Sigma$  on  $(\check{\mathbb{G}}, \check{\mathbb{T}})$  such that the composition  $\Sigma \rightarrow \mathrm{Aut}(\check{\mathbb{G}}, \check{\mathbb{T}}) \rightarrow \mathrm{Out}(\mathbb{G})$  coincides with  $\Sigma \rightarrow \mathrm{Aut}(\mathbb{G}, \mathbb{T}) \rightarrow \mathrm{Out}(\mathbb{G})$ , we form the semi-direct product  $G^L := \check{\mathbb{G}} \ltimes \Sigma$  included into an exact sequence  $1 \rightarrow \check{\mathbb{G}} \rightarrow \check{\mathbb{G}} \ltimes \Sigma \rightarrow \Sigma \rightarrow 1$ . This is a version of the  $L$ -group associated to  $G_F$ . Here  $G_F$  denotes the restriction of the group scheme  $G$  to the generic point  $\mathrm{Spec} F \in X$  of  $X$  (cf. [5]).

**2.3 THETA-LIFTING FUNCTORS** The following notations are borrowed from [5]. Write  $\Omega$  for the canonical line bundle on  $X$ . For  $k \geq 1$  let  $G_k$  denote the sheaf of automorphisms of  $\mathcal{O}_X^k \oplus \Omega^k$  preserving the natural symplectic form  $\wedge^2(\mathcal{O}_X^k \oplus \Omega^k) \rightarrow \Omega$ . The stack  $\mathrm{Bun}_{G_k}$  of  $G_k$ -torsors on  $X$  classifies  $M \in \mathrm{Bun}_{2k}$  equipped with a symplectic form  $\wedge^2 M \rightarrow \Omega$ . Write  $\mathcal{A}_{G_k}$  for the line bundle on  $\mathrm{Bun}_{G_k}$  with fibre  $\det \mathrm{R}\Gamma(X, M)$  at  $M$ , we view it as  $\mathbb{Z}/2\mathbb{Z}$ -graded of parity zero. Let  $\widetilde{\mathrm{Bun}}_{G_k} \rightarrow \mathrm{Bun}_{G_k}$  denote the  $\mu_2$ -gerb of square roots of  $\mathcal{A}_{G_k}$ . Write  $\mathrm{Aut}$  for the perverse theta-sheaf on  $\widetilde{\mathrm{Bun}}_{G_k}$  (cf. also [6]).

Let  $n, m \in \mathbb{N}$  and  $\mathbb{G} = G = \mathrm{GSp}_{2n}$ . Pick a maximal torus and Borel subgroup  $\mathbb{T}_{\mathbb{G}} \subset \mathbb{B}_{\mathbb{G}} \subset \mathbb{G}$ . The stack  $\mathrm{Bun}_G$  classifies  $M \in \mathrm{Bun}_{2n}, \mathcal{A} \in \mathrm{Bun}_1$  with symplectic form  $\wedge^2 M \rightarrow \mathcal{A}$ . Write  $\mathcal{A}_G$  for the  $\mathbb{Z}/2\mathbb{Z}$ -graded line bundle on  $\mathrm{Bun}_G$  with fibre  $\det \mathrm{R}\Gamma(X, M)$  at  $(M, \mathcal{A})$ .

Write  $\tilde{\omega}_0$  for the character of  $\mathbb{G}$  such that  $\mathcal{A}$  is obtained from  $(M, \mathcal{A})$  by the extension of scalars  $\tilde{\omega}_0 : G \rightarrow \mathbb{G}_m$ . Write  ${}_a\mathrm{Sph}_{\mathbb{G}} \subset \mathrm{Sph}_{\mathbb{G}}$  for the full subcategory of objects that vanish on the connected components  $\mathrm{Gr}_{\mathbb{G}}^{\theta}$  satisfying  $\langle \theta, \tilde{\omega}_0 \rangle = -a$ .

Let  $\pi : \tilde{X} \rightarrow X$  be an étale degree 2 covering with Galois group  $\Sigma = \{\mathrm{id}, \sigma\}$ . Let  $\mathcal{E}$  be the  $\sigma$ -anti-invariants in  $\pi_*\mathcal{O}$ , it is equipped with a trivialization  $\mathcal{E}^2 \xrightarrow{\sim} \mathcal{O}_X$ .

Let  $\mathbb{H} = \mathrm{GO}_{2m}^0$  be the connected component of unity of the split orthogonal similitude group  $\mathrm{GO}_{2m}$  over  $k$ . Pick a maximal torus and Borel subgroup  $\mathbb{T}_{\mathbb{H}} \subset \mathbb{B}_{\mathbb{H}} \subset \mathbb{H}$ . Pick  $\tilde{\sigma} \in \mathbb{O}_{2m}(k)$  with  $\tilde{\sigma}^2 = 1$  such that  $\tilde{\sigma} \notin \mathrm{SO}_{2m}(k)$ . We assume in addition that  $\tilde{\sigma}$  preserves  $\mathbb{T}_{\mathbb{H}}$  and  $\mathbb{B}_{\mathbb{H}}$ , so for  $m \geq 2$  it induces the unique<sup>1</sup> nontrivial automorphism of the Dynkin diagram of  $\mathbb{H}$ . For  $m = 1$  we identify  $\mathbb{H} \xrightarrow{\sim} \mathbb{G}_m \times \mathbb{G}_m$  in such a way that  $\tilde{\sigma}$  permutes the two copies of  $\mathbb{G}_m$ .

Realize  $\mathbb{H}$  as the subgroup of  $\mathrm{GL}_{2m}$  preserving up to a multiple the symmetric form given by the matrix

$$\begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix},$$

where  $E_m \in \mathrm{GL}_m$  is the unity. Take  $\mathbb{T}_H$  to be the maximal torus of diagonal matrices,  $\mathbb{B}_H$  the Borel subgroup preserving for  $i = 1, \dots, m$  the isotropic subspace generated by the first  $i$  base

<sup>1</sup>except for  $m = 4$ . The group  $\mathrm{GO}_8$  also has trilinear outer forms, we do not consider them.

vectors  $\{e_1, \dots, e_i\}$ . Then one may take  $\tilde{\sigma}$  interchanging  $e_m$  and  $e_{2m}$  and acting trivially on the orthogonal complement to  $\{e_m, e_{2m}\}$ .

Consider the corresponding  $\Sigma$ -action on  $\mathbb{H}$  by conjugation. Let  $\tilde{H}$  be the group scheme on  $X$ , the twisting of  $\mathbb{H}$  by the  $\Sigma$ -torsor  $\pi : \tilde{X} \rightarrow X$ .

The stack  $\text{Bun}_{\tilde{H}}$  classifies:  $V \in \text{Bun}_{2m}$ ,  $\mathcal{C} \in \text{Bun}_1$ , a nondegenerate symmetric form  $\text{Sym}^2 V \rightarrow \mathcal{C}$ , and a compatible trivialization  $\gamma : \mathcal{C}^{-m} \otimes \det V \xrightarrow{\sim} \mathcal{E}$ . This means that the composition

$$\mathcal{C}^{-2m} \otimes (\det V)^2 \xrightarrow{\gamma^2} \mathcal{E}^2 \xrightarrow{\sim} \mathcal{O}$$

is the isomorphism induced by  $V \xrightarrow{\sim} V^* \otimes \mathcal{C}$ .

Write  $\check{\alpha}_0$  for the character of  $\mathbb{H}$  such that  $\mathcal{C}$  is the extension of scalars of  $(V, \mathcal{C})$  under  $\check{\alpha}_0 : \mathbb{H} \rightarrow \mathbb{G}_m$ . Write  ${}_a \text{Sph}_{\mathbb{H}} \subset \text{Sph}_{\mathbb{H}}$  for the full subcategory of objects that vanish off the connected components  $\text{Gr}_{\mathbb{H}}^\theta$  of  $\text{Gr}_{\mathbb{H}}$  satisfying  $\langle \theta, \check{\alpha}_0 \rangle = -a$ .

Let  $\text{RCov}^0$  denote the stack classifying a line bundle  $\mathcal{U}$  on  $X$  together with a trivialization  $\mathcal{U}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}$ . Its connected components are indexed by  $H_{\text{et}}^1(X, \mathbb{Z}/2\mathbb{Z})$ , each connected component is isomorphic to the classifying stack  $B(\mu_2)$ .

Let  $\text{Bun}_H$  be the stack classifying  $V \in \text{Bun}_{2m}$ ,  $\mathcal{C} \in \text{Bun}_1$  and a symmetric form  $\text{Sym}^2 V \rightarrow \mathcal{C}$  such that the corresponding trivialization  $(\mathcal{C}^{-m} \otimes \det V)^2 \xrightarrow{\sim} \mathcal{O}$  lies in the component of  $\text{RCov}^0$  given by  $(\mathcal{E}, \kappa)$ . Note that

$$\text{Bun}_{\tilde{H}} \xrightarrow{\sim} \text{Spec } k \times_{\text{RCov}^0} \text{Bun}_H,$$

where the map  $\text{Spec } k \rightarrow \text{RCov}^0$  is given by  $(\mathcal{E}, \kappa)$ . Write  $\rho_H : \text{Bun}_{\tilde{H}} \rightarrow \text{Bun}_H$  for the projection.

Let  $\mathcal{A}_H$  be the  $\mathbb{Z}/2\mathbb{Z}$ -graded line bundle on  $\text{Bun}_H$  with fibre  $\det \text{R}\Gamma(X, V)$  at  $(V, \mathcal{C})$ . Set

$$\text{Bun}_{G,H} = \text{Bun}_H \times_{\text{Pic } X} \text{Bun}_G,$$

where the map  $\text{Bun}_H \rightarrow \text{Pic } X$  sends  $(V, \mathcal{C}, \text{Sym}^2 V \rightarrow \mathcal{C})$  to  $\Omega \otimes \mathcal{C}^{-1}$ , and  $\text{Bun}_G \rightarrow \text{Pic } X$  sends  $(M, \wedge^2 M \rightarrow \mathcal{A})$  to  $\mathcal{A}$ . So, we have an isomorphism  $\mathcal{C} \otimes \mathcal{A} \xrightarrow{\sim} \Omega$  for a point of  $\text{Bun}_{G,H}$ . Write  $\text{Bun}_{G,\tilde{H}}$  for the stack obtained from  $\text{Bun}_{G,H}$  by the base change  $\text{Bun}_{\tilde{H}} \rightarrow \text{Bun}_H$ . Let

$$\tau : \text{Bun}_{G,H} \rightarrow \text{Bun}_{G_{2nm}}$$

be the map sending a point as above to  $V \otimes M$  with the induced symplectic form  $\wedge^2(V \otimes M) \rightarrow \Omega$ .

By ([5], Proposition 2), for a point  $(M, \mathcal{A}, V, \mathcal{C})$  of  $\text{Bun}_{G,H}$  there is a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\det \text{R}\Gamma(X, V \otimes M) \xrightarrow{\sim} \frac{\det \text{R}\Gamma(X, V)^{2n} \otimes \det \text{R}\Gamma(X, M)^{2m}}{\det \text{R}\Gamma(X, \mathcal{O})^{2nm} \otimes \det \text{R}\Gamma(X, \mathcal{A})^{2nm}} \quad (2)$$

It yields a map  $\tilde{\tau} : \text{Bun}_{G,H} \rightarrow \widetilde{\text{Bun}}_{G_{2nm}}$  sending  $(\wedge^2 M \rightarrow \mathcal{A}, \text{Sym}^2 V \rightarrow \mathcal{C}, \mathcal{A} \otimes \mathcal{C} \xrightarrow{\sim} \Omega)$  to  $(\wedge^2(M \otimes V) \rightarrow \Omega, \mathcal{B})$ . Here

$$\mathcal{B} = \frac{\det \text{R}\Gamma(X, V)^n \otimes \det \text{R}\Gamma(X, M)^m}{\det \text{R}\Gamma(X, \mathcal{O})^{nm} \otimes \det \text{R}\Gamma(X, \mathcal{A})^{nm}},$$

and  $\mathcal{B}^2$  is identified with  $\det \text{R}\Gamma(X, M \otimes V)$  via (2).

**Definition 1.** Set  $\text{Aut}_{G,H} = \tilde{\tau}^* \text{Aut}[\dim, \text{rel}(\tau)]$ . For the diagram of projections

$$\text{Bun}_H \xleftarrow{\mathfrak{q}} \text{Bun}_{G,H} \xrightarrow{\mathfrak{p}} \text{Bun}_G$$

define  $F_G : D(\text{Bun}_H) \rightarrow D(\text{Bun}_G)$  and  $F_H : D(\text{Bun}_G) \rightarrow D(\text{Bun}_H)$  by

$$F_G(K) = \mathfrak{p}_!(\text{Aut}_{G,H} \otimes \mathfrak{q}^* K)[- \dim \text{Bun}_H]$$

$$F_H(K) = \mathfrak{q}_!(\text{Aut}_{G,H} \otimes \mathfrak{p}^* K)[- \dim \text{Bun}_G]$$

Since  $\mathfrak{p}$  and  $\mathfrak{q}$  are not representable,  $F_G$  and  $F_H$  a priori may send a bounded complex to a complex, which is not bounded even over some open substack of finite type. Let also  $F_{\tilde{H}}$  denote  $F_H$  followed by restriction under  $\text{Bun}_{\tilde{H}} \rightarrow \text{Bun}_H$ . Write  $\text{Aut}_{G,\tilde{H}}$  for the restriction of  $\text{Aut}_{G,H}$  under  $\text{Bun}_{G,\tilde{H}} \rightarrow \text{Bun}_{G,H}$ . By abuse of notation, the composition  $F_G \circ (\rho_H)_!$  is also denoted  $F_G$ .

**2.4 MORPHISM OF L-GROUPS** For  $m \geq 2$  let  $i_{\mathbb{H}} \in \text{Spin}_{2m}$  be the central element of order 2 such that  $\text{Spin}_{2m}/\{i_{\mathbb{H}}\} \xrightarrow{\sim} \text{SO}_{2m}$ . Here  $\text{Spin}_{2m}$  and  $\text{SO}_{2m}$  denote the corresponding split groups over  $\text{Spec } k$ . For  $m \geq 2$  denote by  $\text{GSpin}_{2m}$  the quotient of  $\mathbb{G}_m \times \text{Spin}_{2m}$  by the subgroup generated by  $(-1, i_{\mathbb{H}})$ . Let us convent that  $\text{GSpin}_2 \xrightarrow{\sim} \mathbb{G}_m \times \mathbb{G}_m$ . The Langlands dual group is  $\check{\mathbb{H}} \xrightarrow{\sim} \text{GSpin}_{2m}$ . We also have  $\check{\mathbb{G}} \xrightarrow{\sim} \text{GSpin}_{2n+1}$ , where  $\text{GSpin}_{2n+1} := \mathbb{G}_m \times \text{Spin}_{2n+1}/\{(-1, i_{\mathbb{G}})\}$ . Here  $i_{\mathbb{G}} \in \text{Spin}_{2n+1}$  is the nontrivial central element.

Let  $V_{\mathbb{H}}$  (resp.,  $V_{\mathbb{G}}$ ) denote the standard representation of  $\text{SO}_{2m}$  (resp., of  $\text{SO}_{2n+1}$ ).

**CASE  $m \leq n$ .** Pick an inclusion  $V_{\mathbb{H}} \hookrightarrow V_{\mathbb{G}}$  compatible with symmetric forms. It yields an inclusion  $\check{\mathbb{H}} \hookrightarrow \check{\mathbb{G}}$ , which we assume compatible with the corresponding maximal tori. Pick an element  $\sigma_{\mathbb{G}} \in \text{SO}(V_{\mathbb{G}}) \xrightarrow{\sim} \check{\mathbb{G}}_{ad}$  normalizing  $\check{\mathbb{T}}_{\mathbb{G}}$  and preserving  $V_{\mathbb{H}}$  and  $\check{\mathbb{T}}_{\mathbb{H}} \subset \check{\mathbb{B}}_{\mathbb{H}}$ . Let  $\sigma_{\mathbb{H}} \in \mathcal{O}(V_{\mathbb{H}})$  be its restriction to  $V_{\mathbb{H}}$ . We assume that  $\sigma_{\mathbb{H}}$  viewed as an automorphism of  $(\check{\mathbb{H}}, \check{\mathbb{T}}_{\mathbb{H}})$  extends the action of  $\Sigma$  on the roots datum of  $(\check{\mathbb{H}}, \check{\mathbb{T}}_{\mathbb{H}})$  defined in Section 2.3.

In concrete terms, one may take  $V_{\mathbb{G}} = \mathbb{Q}_{\ell}^{2n+1}$  with symmetric form given by the matrix

$$\begin{pmatrix} 0 & E_n & 0 \\ E_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where  $E_n \in \text{GL}_n$  is the unity. Take  $\check{\mathbb{T}}_{\mathbb{G}}$  to be the preimage of the torus of diagonal matrices under  $\check{\mathbb{G}} \rightarrow \text{SO}_{2n+1}$ . Let  $V_{\mathbb{H}} \subset V_{\mathbb{G}}$  be generated by  $\{e_1, \dots, e_m, e_{n+1}, \dots, e_{n+m}\}$ . Let  $\check{\mathbb{T}}_{\mathbb{H}}$  be the preimage under  $\check{\mathbb{H}} \rightarrow \text{SO}(V_{\mathbb{H}})$  of the torus of diagonal matrices, and  $\check{\mathbb{B}}_{\mathbb{H}}$  the Borel subgroup preserving for  $i = 1, \dots, m$  the isotropic subspace generated by  $\{e_1, \dots, e_i\}$ . Then one may take  $\sigma_{\mathbb{G}}$  permuting  $e_m$  and  $e_{n+m}$ , sending  $e_{2n+1}$  to  $-e_{2n+1}$  and acting trivially on the other base vectors.

We let  $\Sigma$  act on  $\check{\mathbb{H}}$  and  $\check{\mathbb{G}}$  via the elements  $\sigma_{\mathbb{H}}, \sigma_{\mathbb{G}}$ . So, the inclusion  $\check{\mathbb{H}} \hookrightarrow \check{\mathbb{G}}$  is  $\Sigma$ -equivariant and yields a morphism of the  $L$ -groups  $\tilde{H}^L \rightarrow G^L$ .

**CASE  $m > n$ .** Pick an inclusion  $V_{\mathbb{G}} \hookrightarrow V_{\mathbb{H}}$  compatible with symmetric forms. It yields an inclusion  $\check{\mathbb{G}} \hookrightarrow \check{\mathbb{H}}$ , which we assume compatible with the corresponding maximal tori. Let  $\sigma_{\mathbb{G}}$

be the identical automorphism of  $V_{\mathbb{G}}$ . Extend it to an element  $\sigma_{\mathbb{H}} \in \mathbb{O}(V_{\mathbb{H}})$  by requiring that  $\sigma_{\mathbb{H}}$  preserves  $\check{\mathbb{T}}_{\mathbb{H}} \subset \check{\mathbb{B}}_{\mathbb{H}}$  and  $\sigma_{\mathbb{H}} \notin \mathrm{SO}(V_{\mathbb{H}})$ ,  $\sigma_{\mathbb{H}}^2 = \mathrm{id}$ .

In concrete terms, take the symmetric form on  $V_{\mathbb{H}} = \bar{\mathbb{Q}}_{\ell}^{2m}$  given by the matrix

$$\begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix}$$

Let  $V_{\mathbb{G}}$  be the subspace of  $V_{\mathbb{H}}$  generated by  $\{e_1, \dots, e_n; e_{m+1}, \dots, e_{m+n}; e_{n+1} + e_{m+n+1}\}$ . Take  $\check{\mathbb{T}}_{\mathbb{H}}$  to be the preimage under  $\check{\mathbb{H}} \rightarrow \mathrm{SO}(V_{\mathbb{H}})$  of the torus of diagonal matrices, and  $\check{\mathbb{B}}_{\mathbb{H}}$  the Borel subgroup preserving for  $i = 1, \dots, m$  the isotropic subspace of  $V_{\mathbb{H}}$  generated by  $\{e_1, \dots, e_i\}$ . Let  $\check{\mathbb{T}}_{\mathbb{G}}$  be the preimage under  $\check{\mathbb{G}} \rightarrow \check{\mathbb{H}}$  of  $\check{\mathbb{T}}_{\mathbb{H}}$ . Let  $\sigma_{\mathbb{H}} \in \mathbb{O}(V_{\mathbb{H}})$  permute  $e_m$  and  $e_{2m}$  and act trivially on the orthogonal complement to  $\{e_m, e_{2m}\}$ . Then  $\sigma_{\mathbb{H}}$  lifts uniquely to an automorphism of the exact sequence  $1 \rightarrow \mathbb{G}_m \rightarrow \check{\mathbb{H}} \rightarrow \mathrm{SO}(V_{\mathbb{H}}) \rightarrow 1$  that acts trivially on  $\mathbb{G}_m$ .

As above, the inclusion  $\check{\mathbb{G}} \hookrightarrow \check{\mathbb{H}}$  is  $\Sigma$ -equivariant and gives rise to a morphism of the L-groups  $\check{\mathbb{G}} \times \Sigma = G^L \rightarrow \check{H}^L$ .

**Theorem 1.** 1) For  $m \leq n$  there is a homomorphism  $\kappa : \check{\mathbb{H}} \times \mathbb{G}_m \rightarrow \check{\mathbb{G}}$  with the following property. There exists an isomorphism

$$(\pi \times \mathrm{id})^* \mathrm{H}_G^-(\mathcal{S}, F_G(K)) \xrightarrow{\sim} (\mathrm{id} \boxtimes F_G)(\mathrm{H}_H^-(\mathrm{gRes}^{\kappa}(\mathcal{S}), K)) \quad (3)$$

in  $\mathrm{D}(\tilde{X} \times \mathrm{Bun}_G)$  functorial in  $\mathcal{S} \in \mathrm{Sph}_{\mathbb{G}}$  and  $K \in \mathrm{D}(\mathrm{Bun}_{\check{H}})$ . Here  $\pi \times \mathrm{id} : \tilde{X} \times \mathrm{Bun}_G \rightarrow X \times \mathrm{Bun}_G$ , and  $\mathrm{id} \boxtimes F_G : \mathrm{D}(\tilde{X} \times \mathrm{Bun}_{\check{H}}) \rightarrow \mathrm{D}(\tilde{X} \times \mathrm{Bun}_G)$  is the corresponding theta-lifting functor.

2) For  $m > n$  there is a homomorphism  $\kappa : \check{\mathbb{G}} \times \mathbb{G}_m \rightarrow \check{\mathbb{H}}$  with the following property. There exists an isomorphism

$$\mathrm{H}_{\check{H}}^-(\mathcal{S}, F_{\check{H}}(K)) \xrightarrow{\sim} (\pi \times \mathrm{id})^*(\mathrm{id} \boxtimes F_{\check{H}})(\mathrm{H}_{\check{\mathbb{G}}}^-(\mathrm{gRes}^{\kappa}(*\mathcal{S}), K))$$

in  $\mathrm{D}(\tilde{X} \times \mathrm{Bun}_{\check{H}})$  functorial in  $\mathcal{S} \in \mathrm{Sph}_{\mathbb{H}}$  and  $K \in \mathrm{D}(\mathrm{Bun}_G)$ . Here  $\pi \times \mathrm{id} : \tilde{X} \times \mathrm{Bun}_{\check{H}} \rightarrow X \times \mathrm{Bun}_{\check{H}}$  and  $\mathrm{id} \boxtimes F_{\check{H}} : \mathrm{D}(X \times \mathrm{Bun}_G) \rightarrow \mathrm{D}(X \times \mathrm{Bun}_{\check{H}})$  is the corresponding theta-lifting functor.

*Remark 1.* If  $m = n$  or  $m = n + 1$  then the restriction of  $\kappa$  to  $\mathbb{G}_m$  is trivial. The explicit formulas for  $\kappa$  are given in Section 4.8.9. If  $m \leq n$  then  $\kappa$  fits into the diagram

$$\begin{array}{ccc} \check{\mathbb{H}} \times \mathbb{G}_m & \xrightarrow{\kappa} & \check{\mathbb{G}} \\ \downarrow & & \downarrow \\ \mathrm{SO}_{2m} \times \mathbb{G}_m & \xrightarrow{\bar{\kappa}} & \mathrm{SO}_{2n+1} \end{array}$$

If  $m > n$  then  $\kappa$  fits into the diagram

$$\begin{array}{ccc} \check{\mathbb{G}} \times \mathbb{G}_m & \xrightarrow{\kappa} & \check{\mathbb{H}} \\ \downarrow & & \downarrow \\ \mathrm{SO}_{2n+1} \times \mathbb{G}_m & \xrightarrow{\bar{\kappa}} & \mathrm{SO}_{2m}, \end{array}$$

In both cases  $\bar{\kappa}$  is the map from ([7], Theorem 3).



For  $a \in \mathbb{Z}$  let  ${}^a\text{Bun}_{G,\tilde{H}}$  be the stack classifying  $\tilde{x} \in \tilde{X}$ ,  $(M, \mathcal{A}) \in \text{Bun}_G$ ,  $(V, \mathcal{C}, \gamma) \in \text{Bun}_{\tilde{H}}$ , and an isomorphism  $\mathcal{A} \otimes \mathcal{C} \xrightarrow{\sim} \Omega(a\pi(\tilde{x}))$ . We have the Hecke functors defined as in Section 2.2

$$H_G^\leftarrow : {}_{-a}\text{Sph}_{\mathbb{G}} \times D(\text{Bun}_{G,\tilde{H}}) \rightarrow D({}^a\text{Bun}_{G,\tilde{H}})$$

and

$$H_{\tilde{H}}^\leftarrow : {}_{-a}\text{Sph}_{\mathbb{H}} \times D(\text{Bun}_{G,\tilde{H}}) \rightarrow D({}^a\text{Bun}_{G,\tilde{H}})$$

Set also  $H_{\tilde{H}}^\rightarrow(\mathcal{S}, \cdot) = H_{\tilde{H}}^\leftarrow(*\mathcal{S}, \cdot)$ . We will derive Theorem 1 from the following Hecke property of  $\text{Aut}_{G,\tilde{H}}$ .

**Theorem 2.** *Let  $\kappa$  be as in Theorem 1.*

1) *For  $m \leq n$  there exists an isomorphism*

$$H_G^\leftarrow(\mathcal{S}, \text{Aut}_{G,\tilde{H}}) \xrightarrow{\sim} H_{\tilde{H}}^\leftarrow(*\text{gRes}^\kappa(\mathcal{S}), \text{Aut}_{G,\tilde{H}}) \quad (4)$$

*in  $D({}^a\text{Bun}_{G,\tilde{H}})$  functorial in  $\mathcal{S} \in {}_{-a}\text{Sph}_{\mathbb{G}}$ .*

2) *For  $m > n$  there exists an isomorphism*

$$H_{\tilde{H}}^\leftarrow(\mathcal{S}, \text{Aut}_{G,\tilde{H}}) \xrightarrow{\sim} H_G^\leftarrow(\text{gRes}^\kappa(*\mathcal{S}), \text{Aut}_{G,\tilde{H}}) \quad (5)$$

*in  $D({}^a\text{Bun}_{G,\tilde{H}})$  functorial in  $\mathcal{S} \in {}_{-a}\text{Sph}_{\mathbb{H}}$ .*

## 2.5 APPLICATION: AUTOMORPHIC SHEAVES ON $\text{Bun}_{\text{GSp}_4}$ .

Keep the notation of Section 2.3 assuming  $m = n = 2$ , so  $G = \text{GSp}_4$ . Let  $\tilde{E}$  be an irreducible rank two smooth  $\mathbb{Q}_\ell$ -sheaf on  $\tilde{X}$ ,  $\chi$  a rank one local system on  $X$  equipped with an isomorphism  $\pi^*\chi \xrightarrow{\sim} \det \tilde{E}$ . To this data one associates the perverse sheaf  $K_{\tilde{E},\chi,\tilde{H}}$  on  $\text{Bun}_{\tilde{H}}$  introduced in ([7], Section 5.1). The local system  $\pi_*\tilde{E}^*$  is equipped with a natural symplectic form  $\wedge^2(\pi_*\tilde{E}^*) \rightarrow \chi^{-1}$ , so gives rise to a  $\check{G}$ -local system  $E_{\check{G}}$  on  $X$ . Since  $K_{\tilde{E},\chi,\tilde{H}}$  is a Hecke eigensheaf, Theorem 1 implies the following.

**Corollary 1.** *The complex  $F_G(\rho_H!K_{\tilde{E},\chi,\tilde{H}}) \in D(\text{Bun}_G)$  is a Hecke eigensheaf corresponding to the  $\check{G}$ -local system  $E_{\check{G}}$ .*

*Remark 2.* i) We expect that for each open substack of finite type  $\mathcal{U} \subset \text{Bun}_G$  the restriction of  $F_G(\rho_H!K_{\tilde{E},\chi,\tilde{H}})$  to  $\mathcal{U}$  is a bounded complex. We also expect it to be perverse.

ii) If  $\tilde{X}$  splits fix a numbering of connected components of  $\tilde{X}$ . Then  $\tilde{E}$  becomes a pair of irreducible rank 2 local systems  $E_1, E_2$  on  $X$  with the isomorphisms  $\det E_1 \xrightarrow{\sim} \det E_2 \xrightarrow{\sim} \chi$ .

## 3. LOCAL THEORY

### 3.1 BACKGROUND ON NON-RAMIFIED WEIL CATEGORY

Remind the following constructions from [8]. Let  $W$  be a symplectic Tate space over  $k$ . By definition ([2], 4.2.13),  $W$  is a complete topological  $k$ -vector space having a base of neighbourhoods

of 0 consisting of commesurable vector subspaces (i.e.,  $\dim U_1/(U_1 \cap U_2) < \infty$  for any  $U_1, U_2$  from this base). It is equipped with a (continuous) symplectic form  $\wedge^2 W \rightarrow k$  (it induces a topological isomorphism  $W \xrightarrow{\sim} W^*$ ).

For a  $k$ -subspace  $L \subset W$  write  $L^\perp = \{w \in W \mid \langle w, l \rangle = 0 \text{ for all } l \in L\}$ . Write  $\mathcal{L}_d(W)$  for the scheme of discrete lagrangian lattices in  $W$ . For a c-lattice  $R \subset W$  let  $\mathcal{L}_d(W)_R \subset \mathcal{L}_d(W)$  be the open subscheme of  $L \in \mathcal{L}_d(W)$  satisfying  $L \cap R = 0$ .

For a  $k$ -point  $L \in \mathcal{L}_d(W)$  one defines the category  $\mathcal{H}_L$  as in ([8], Section 6.1). Let us remind the definition. For a c-lattice  $R \subset R^\perp \subset W$  with  $R \cap L = 0$  we have a lagrangian subspace  $L_R := L \cap R^\perp \in \mathcal{L}(R^\perp/R)$  and the Heisenberg group  $H_R = R^\perp/R \oplus k$ . Let  $\mathcal{H}_{L_R}$  be the category of perverse sheaves on  $H_R$ , which are  $(\bar{L}_R, \chi_{L,R})$ -equivariant under the left multiplication on  $H_R$ . Here  $\bar{L}_R = L_R \times \mathbb{A}^1 \subset H_R$  and  $\chi_{L,R}$  is the local system  $\text{pr}^* \mathcal{L}_\psi$  for the projection  $\text{pr} : \bar{L}_R \rightarrow \mathbb{A}^1$  sending  $(l, a)$  to  $a$ . Let  $D\mathcal{H}_{L_R} \subset D(H_R)$  be the full subcategory of objects whose all perverse cohomologies lie in  $\mathcal{H}_{L_R}$ .

For another c-lattice  $S \subset R$  we have (an exact for the perverse t-structures) transition functor  $T_{S,R}^L : D\mathcal{H}_{L_R} \rightarrow D\mathcal{H}_{L_S}$  (cf. *loc.cit.*, Section 6.1). Now  $\mathcal{H}_L$  is the inductive 2-limit of  $\mathcal{H}_{L_R}$  over the partially ordered set of c-lattices  $R \subset R^\perp$  such that  $R \cap L = 0$ .

Given a c-lattice  $M$  in  $W$ , we have a  $\mathbb{Z}/2\mathbb{Z}$ -graded line bundle on  $\mathcal{L}_d(W)$ , whose fibre at  $L$  is  $\det(M : L)$ . Remind that

$$\det(M : L) = \det(M \oplus L \rightarrow W),$$

where the complex  $M \oplus L \rightarrow W$  is placed in cohomological degrees 0 and 1. If  $S \subset M \subset S^\perp$  is a c-lattice with  $S \cap L = 0$  then  $\det(M : L) \xrightarrow{\sim} \det(M/S) \otimes \det L_S$ , where  $L_S := L \cap S^\perp$ . Note that  $\det(M : L) \xrightarrow{\sim} \det(M^\perp : L)$  canonically. If  $M' \subset W$  is another c-lattice then we have  $\det(M : L) \xrightarrow{\sim} \det(M : M') \otimes \det(M' : L)$  canonically. If  $R' \subset W$  is a lagrangian c-lattice then, as  $\mathbb{Z}/2\mathbb{Z}$ -graded,  $\det(M : L)$  is of parity  $\dim(R' : M) \bmod 2$ .

Fix a one-dimensional  $\mathbb{Z}/2\mathbb{Z}$ -graded space  $\mathcal{J}_W$  placed in degree  $\dim(R' : M) \bmod 2$ . Let  $\mathcal{A}_d$  be the  $\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero line bundle on  $\mathcal{L}_d(W)$  with fibre  $\mathcal{J}_W \otimes \det(M : L)$  at  $L$ . Let  $\tilde{\mathcal{L}}_d(W)$  be the  $\mu_2$ -gerb of square roots of  $\mathcal{A}_d$ .

For  $k$ -points  $N^0, L^0 \in \tilde{\mathcal{L}}_d(W)$  one associates to them in a canonical way a functor  $\mathcal{F}_{N^0, L^0} : D\mathcal{H}_L \rightarrow D\mathcal{H}_N$  sending  $\mathcal{H}_L$  to  $\mathcal{H}_N$  (defined as in [8], Section 6.2). Let us precise some details. For a c-lattice  $R \subset R^\perp$  in  $W$  we have the projection

$$\mathcal{L}_d(W)_R \rightarrow \mathcal{L}(R^\perp/R)$$

sending  $L$  to  $L_R$ . Let  $\mathcal{A}_R$  be the  $\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero line bundle on  $\mathcal{L}(R^\perp/R)$  whose fibre at  $L_1$  is  $\det L_1 \otimes \det(M : R) \otimes \mathcal{J}_W$ . Its restriction to  $\mathcal{L}_d(W)_R$  identifies canonically with  $\mathcal{A}_d$ , hence a morphism of stacks

$$\tilde{\mathcal{L}}_d(W)_R \rightarrow \tilde{\mathcal{L}}(R^\perp/R) \tag{6}$$

where  $\tilde{\mathcal{L}}(R^\perp/R)$  is the gerb of square roots of  $\mathcal{A}_R$ . Write  $N_R^0, L_R^0$  for the images of  $N^0, L^0$  under (6). By definition, the enhanced structure on  $L_R$  and  $N_R$  is given by one-dimensional spaces

$\mathcal{B}_L, \mathcal{B}_N$  equipped with

$$\mathcal{B}_L^2 \xrightarrow{\sim} \det L_R \otimes \det(M : R) \otimes \mathcal{J}_W, \quad \mathcal{B}_N^2 \xrightarrow{\sim} \det N_R \otimes \det(M : R) \otimes \mathcal{J}_W,$$

hence an isomorphism  $\mathcal{B}^2 \xrightarrow{\sim} \det L_R \otimes \det N_R$  for  $\mathcal{B} := \mathcal{B}_L \otimes \mathcal{B}_N \otimes \det(M : R)^{-1} \otimes \mathcal{J}_W^{-1}$ . Write

$$\mathcal{F}_{N_R^0, L_R^0} : D\mathcal{H}_{L_R} \rightarrow D\mathcal{H}_{N_R} \quad (7)$$

for the canonical intertwining functor corresponding to  $(N_R, L_R, \mathcal{B})$  (as in *loc.cit*, Section 6.2). Then  $\mathcal{F}_{N^0, L^0}$  is defined as the limit of the functors (7) over the partially ordered set of c-lattices  $R \subset R^\perp$  such that  $N, R \in \mathcal{L}_d(W)_R$ .

The proof of (Theorem 2, [8]) holds through, so for a  $k$ -point  $L^0 \in \tilde{\mathcal{L}}_d(W)$  we have the functor  $\mathcal{F}_{L^0} : D\mathcal{H}_L \rightarrow D(\tilde{\mathcal{L}}_d(W))$  exact for the perverse t-structures. For two  $k$ -points  $L^0, N^0 \in \tilde{\mathcal{L}}_d(W)$  the diagram is canonically 2-commutative

$$\begin{array}{ccc} D\mathcal{H}_L & \xrightarrow{\mathcal{F}_{L^0}} & D(\tilde{\mathcal{L}}_d(W)) \\ \downarrow \mathcal{F}_{N^0, L^0} & \nearrow \mathcal{F}_{N^0} & \\ D\mathcal{H}_N & & \end{array}$$

The non-ramified Weil category  $W(\tilde{\mathcal{L}}_d(W))$  is defined as the essential image of  $\mathcal{F}_{L^0} : \mathcal{H}_L \rightarrow P(\tilde{\mathcal{L}}_d(W))$  for any  $k$ -point  $L^0 \in \tilde{\mathcal{L}}_d(W)$ .

3.2 Let  $\mathcal{O}$  be a complete discrete valuation  $k$ -algebra,  $F$  its fraction field. Write  $\Omega$  for the completed module of relative differentials of  $\mathcal{O}$  over  $k$ . For a free  $\mathcal{O}$ -module  $V$  of finite rank write  $V(r) \subset V \otimes F$  for the  $\mathcal{O}$ -submodule  $t^{-r}V$ , where  $t \in \mathcal{O}$  is any uniformizer.

For  $r \in \mathbb{Z}$  let  $W_r$  be a free  $\mathcal{O}$ -module of rank  $2n$  with symplectic form  $\wedge^2 W_r \rightarrow \Omega(r)$ . Then  $W_r(F)$  is a symplectic Tate space with the form  $\wedge^2 W_r(F) \rightarrow \Omega(F) \xrightarrow{\text{Res}} k$ . Set

$$\mathcal{L}_d^{ex} = \sqcup_{r \in \mathbb{Z}} \mathcal{L}_d(W_r(F))$$

Let  $\mathcal{G}_{b,a}$  be the set of  $F$ -linear isomorphisms  $g : W_a(F) \rightarrow W_b(F)$  of symplectic  $F$ -spaces. Let  $G_a = \text{Sp}(W_a)$  as a group scheme over  $\mathcal{O}$ .

Fix a  $\mathbb{Z}/2\mathbb{Z}$ -graded line  $\mathcal{J}_r$  placed in degree  $nr \bmod 2$ . Let  $\mathcal{A}_{d,r}$  be the  $\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero line bundle on  $\mathcal{L}_d(W_r(F))$  whose fibre at  $L$  is  $\mathcal{J}_r \otimes \det(W_r : L)$ . Let  $\tilde{\mathcal{L}}_d(W_r(F))$  be the  $\mu_2$ -gerb of square roots of  $\mathcal{A}_{d,r}$ .

Let  $\tilde{\mathcal{G}}_{b,a}$  be the  $\mu_2$ -gerb over  $\mathcal{G}_{b,a}$  classifying  $g \in \mathcal{G}_{b,a}$ , a one-dimensional space  $\mathcal{B}$  and an isomorphism  $\mathcal{B}^2 \xrightarrow{\sim} \mathcal{J}_b \otimes \mathcal{J}_a^{-1} \otimes \det(W_b : gW_a)$ . The composition  $\mathcal{G}_{c,b} \times \mathcal{G}_{b,a} \rightarrow \mathcal{G}_{c,a}$  lifts to a morphism  $\tilde{\mathcal{G}}_{c,b} \times \tilde{\mathcal{G}}_{b,a} \rightarrow \tilde{\mathcal{G}}_{c,a}$  sending  $(g_2, \mathcal{B}_2) \in \tilde{\mathcal{G}}_{c,b}$ ,  $(g_1, \mathcal{B}_1) \in \tilde{\mathcal{G}}_{b,a}$  to  $(g_2 g_1, \mathcal{B})$ , where  $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$ .

Consider the action map

$$\tilde{\mathcal{G}}_{b,a} \times \tilde{\mathcal{L}}_d(W_a(F)) \rightarrow \tilde{\mathcal{L}}_d(W_b(F))$$

sending  $(g, \mathcal{B}) \in \tilde{\mathcal{G}}_{b,a}$  and  $(L, \mathcal{B}_L) \in \tilde{\mathcal{L}}_d(W_a(F))$  to  $(gL, \mathcal{B}_1)$ , where  $\mathcal{B}_1 = \mathcal{B} \otimes \mathcal{B}_L$  is equipped with the induced isomorphism

$$\mathcal{B}_1^2 \xrightarrow{\sim} \mathcal{J}_b \otimes \det(W_b : gL)$$

In this way  $\tilde{\mathcal{G}}^{ex} := \sqcup_{a,b \in \mathbb{Z}} \tilde{\mathcal{G}}_{b,a}$  becomes a groupoid acting on

$$\tilde{\mathcal{L}}_d^{ex} := \sqcup_{r \in \mathbb{Z}} \tilde{\mathcal{L}}_d(W_r(F))$$

The gerb  $\tilde{\mathcal{G}}_{a,a} \rightarrow \mathcal{G}_{a,a}$  has a canonical section over  $G_a(\mathcal{O}) \subset \mathcal{G}_{a,a}$  sending  $g \in G_a(\mathcal{O})$  to  $(g, \mathcal{B} = k)$  equipped with  $\text{id} : \mathcal{B}^2 \xrightarrow{\sim} \det(W_a : W_a)$ . One can define the equivariant derived category  $D_{G_a(\mathcal{O})}(\tilde{\mathcal{L}}_d(W_a(F)))$  as in ([7], Section 8.2.2).

For  $g \in \mathcal{G}_{b,a}$  and a c-lattice  $R \subset R^\perp \subset W_a(F)$  we have  $(gR)^\perp = g(R^\perp)$ , and  $g$  induces an isomorphism of symplectic spaces

$$g : R^\perp/R \xrightarrow{\sim} (gR)^\perp/(gR) \quad (8)$$

If  $L \in \mathcal{L}_d(W_a(F))_R$  then  $g$  yields an equivalence  $\mathcal{H}_{L_R} \xrightarrow{\sim} \mathcal{H}_{gL_{gR}}$  sending  $K$  to  $g_*K$  for the map  $g : H_R \xrightarrow{\sim} H_{gR}$ . Passing to the limit by  $R$ , we further get an equivalence  $g : \mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_{gL}$ .

**Proposition 1.** *Let  $a, b \in \mathbb{Z}$ ,  $\tilde{g} \in \tilde{\mathcal{G}}_{b,a}$  over  $g \in \mathcal{G}_{b,a}$  and  $L^0 \in \tilde{\mathcal{L}}_d(W_a(F))$  be  $k$ -points. Then the diagram is canonically 2-commutative*

$$\begin{array}{ccc} D\mathcal{H}_L & \xrightarrow{\mathcal{F}_{L^0}} & D(\tilde{\mathcal{L}}_d(W_a(F))) \\ \downarrow g & & \downarrow \tilde{g} \\ D\mathcal{H}_{gL} & \xrightarrow{\mathcal{F}_{\tilde{g}L^0}} & D(\tilde{\mathcal{L}}_d(W_b(F))) \end{array}$$

*Proof* Let  $R \subset R^\perp \subset W_a(F)$  be a c-lattice with  $R \cap L = 0$ . We get an equivalence  $g : \mathcal{H}_{L_R} \xrightarrow{\sim} \mathcal{H}_{gL_{gR}}$ . Let  $\mathcal{A}_R$  be the line bundle on  $\mathcal{L}(R^\perp/R)$  whose fibre at  $L_1$  is

$$\mathcal{J}_a \otimes \det(W_a : R) \otimes \det L_1$$

Let  $\tilde{\mathcal{L}}(R^\perp/R)$  be the  $\mu_2$ -gerb of square roots of  $\mathcal{A}_R$ . We have the projection

$$\tilde{\mathcal{L}}_d(W_a(F))_R \rightarrow \tilde{\mathcal{L}}(R^\perp/R)$$

sending  $L^0$  to  $L_R^0$ . As in ([8], Section 6.4), we have the functors  $\mathcal{F}_{L_R^0} : \mathcal{H}_{L_R} \rightarrow P(\tilde{\mathcal{L}}(R^\perp/R))$ . It suffices to show that the diagram is canonically 2-commutative

$$\begin{array}{ccc} \mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L_R^0}} & P(\tilde{\mathcal{L}}(R^\perp/R)) \\ \downarrow g & & \downarrow \tilde{g} \\ \mathcal{H}_{gL_{gR}} & \xrightarrow{\mathcal{F}_{\tilde{g}L_{gR}^0}} & P(\tilde{\mathcal{L}}((gR)^\perp/gR)) \end{array} \quad (9)$$

The above expression  $\tilde{g}L_{gR}^0$  is the image of  $\tilde{g}(L^0)$  under  $\tilde{\mathcal{L}}_d(W_b(F))_{gR} \rightarrow \tilde{\mathcal{L}}((gR)^\perp/(gR))$ . Note that  $\tilde{g}L_{gR}^0 = \tilde{g}(L_R^0)$ , where

$$\tilde{g} : \tilde{\mathcal{L}}(R^\perp/R) \xrightarrow{\sim} \tilde{\mathcal{L}}((gR)^\perp/gR)$$

sends  $(L_1, \mathcal{B})$  to  $(gL_1, \mathcal{B} \otimes \mathcal{B}_0)$ . Here  $\tilde{g} = (g, \mathcal{B}_0)$ .

Remind that  $H_R$  denotes the Heisenberg group  $R^\perp/R \times \mathbb{A}^1$ . For the isomorphism

$$\tilde{g} : \tilde{\mathcal{L}}(R^\perp/R) \times \tilde{\mathcal{L}}(R^\perp/R) \times H_R \xrightarrow{\sim} \tilde{\mathcal{L}}((gR)^\perp/gR) \times \tilde{\mathcal{L}}((gR)^\perp/gR) \times H_{gR}$$

we have  $\tilde{g}^*F \xrightarrow{\sim} F$  canonically, where  $F$  is the CIO sheaf on each side (introduced in [8], Theorem 1). The 2-commutativity of (9) follows.  $\square$

By Proposition 1, each  $\tilde{g} \in \mathcal{G}_{b,a}$  yields an equivalence  $\tilde{g} : W(\tilde{\mathcal{L}}_d(W_a(F))) \xrightarrow{\sim} W(\tilde{\mathcal{L}}_d(W_b(F)))$ .

3.3 Now assume that we are given for each  $a \in \mathbb{Z}$  a decomposition  $W_a = U_a \oplus U_a^* \otimes \Omega(a)$ , where  $U_a$  is a free  $\mathcal{O}$ -module of rank  $n$ ,  $U_a$  and  $U_a^* \otimes \Omega(a)$  are lagrangians, and the form  $\omega : \wedge^2 W_a \rightarrow \Omega(a)$  is given by  $\omega\langle u, u^* \rangle = \langle u, u^* \rangle$  for  $u \in U_a, u^* \in U_a^* \otimes \Omega(a)$ , where  $\langle \cdot, \cdot \rangle$  is the canonical pairing between  $U_a$  and  $U_a^* \otimes \Omega(a)$ .

*Remark 3.* If  $U_1$  is a free  $\mathcal{O}$ -module of finite rank and  $U_2 \subset U_1(F)$  is a  $\mathcal{O}$ -lattice then there is a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism  $\det(U_2 : U_1)^* \xrightarrow{\sim} \det(U_1^* \otimes \Omega : U_2^* \otimes \Omega)$ .

For  $a, b \in \mathbb{Z}$  let  $\mathcal{U}_{b,a}$  be the set of  $F$ -linear isomorphisms  $U_a(F) \rightarrow U_b(F)$ . We have an inclusion  $\mathcal{U}_{b,a} \hookrightarrow \mathcal{G}_{b,a}$  given by  $g \mapsto (g, ({}^t g)^{-1})$ . Here  ${}^t g \in \mathrm{GL}(U^* \otimes \Omega)(F)$  is the adjoint operator. By Remark 3, for  $g \in \mathcal{U}_{b,a}$  we have canonically

$$\det(W_b : gW_a) \xrightarrow{\sim} \det(U_b : gU_a)^2 \otimes (\det U_{a,x})^a \otimes (\det U_{b,x})^{-b} \otimes \det(\mathcal{O}(-b) : \mathcal{O}(-a))^n [n(b-a)]$$

Assume in addition that  $n$  is even. Assume given a one-dimensional  $\mathbb{Z}/2\mathbb{Z}$ -graded purely of degree zero vector space  $\mathcal{J}_{U,a}$  equipped with  $\mathcal{J}_{U,a}^2 \xrightarrow{\sim} \mathcal{J}_a \otimes \det(U_{a,x})^{-a}$ . This yields a section  $\rho_{b,a} : \mathcal{U}_{b,a} \rightarrow \tilde{\mathcal{G}}_{b,a}$  defined as follows. We send  $g \in \mathcal{U}_{b,a}$  to  $(g, \mathcal{B})$ , where

$$\mathcal{B} = \mathcal{J}_{U,b} \otimes \mathcal{J}_{U,a}^{-1} \otimes \det(U_b : gU_a) \otimes \det(\mathcal{O}(-b) : \mathcal{O}(-a))^{n/2}$$

is equipped with the induced isomorphism

$$\mathcal{B}^2 \xrightarrow{\sim} \mathcal{J}_b \otimes \mathcal{J}_a^{-1} \otimes \det(W_b : gW_a)$$

The section  $\rho$  is compatible with the groupoid structures on  $\tilde{\mathcal{G}}^{ex}$  and  $\mathcal{U}^{ex} = \sqcup_{a,b} \mathcal{U}_{b,a}$ . We let  $\mathcal{U}^{ex}$  act on  $\tilde{\mathcal{L}}_d^{ex}$  via  $\rho$ .

**Proposition 2.** *For  $a \in \mathbb{Z}$  there is a canonical functor  $\mathcal{F}_{U_a(F)} : \mathrm{D}(U_a^* \otimes \Omega(F)) \rightarrow \mathrm{D}(\tilde{\mathcal{L}}_d(W_a(F)))$  exact for the perverse  $t$ -structures. For  $g \in \mathcal{U}_{b,a}$  and  $\tilde{g} = \rho_{b,a}(g)$  the diagram is canonically 2-commutative*

$$\begin{array}{ccc} \mathrm{D}(U_a^* \otimes \Omega(F)) & \xrightarrow{\mathcal{F}_{U_a(F)}} & \mathrm{D}(\tilde{\mathcal{L}}_d(W_a(F))) \\ \downarrow g & & \downarrow \tilde{g} \\ \mathrm{D}(U_b^* \otimes \Omega(F)) & \xrightarrow{\mathcal{F}_{U_b(F)}} & \mathrm{D}(\tilde{\mathcal{L}}_d(W_b(F))), \end{array} \quad (10)$$

*Proof*

**Step 1.** Let  $R_1 \subset R_2 \subset U_a(F)$  be c-lattices. Write  $\langle \cdot, \cdot \rangle_a$  for the symplectic form on the Tate space  $W_a(F)$ . For a c-lattice  $R \subset U_a(F)$  set  $R' = \{w \in U_a^* \otimes \Omega(F) \mid \langle w, r \rangle_a = 0 \text{ for all } r \in R\}$ , this is a c-lattice in  $U_a^* \otimes \Omega(F)$ .

Set  $R = R_1 \oplus R_2'$  then  $R^\perp = R_2 \oplus R_1'$ . Let  $U_R = R_2/R_1$  then  $U_R \in \mathcal{L}(R^\perp/R)$ . Set  $U_R^0 = (U_R, \mathcal{B})$  equipped with the canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\mathcal{B}^2 \xrightarrow{\sim} \mathcal{J}_a \otimes \det(U_R) \otimes \det(W_a : R),$$

where  $\mathcal{B} = \mathcal{J}_{U,a} \otimes \det(U_a : R_1) \otimes \det(\mathcal{O}(-a) : \mathcal{O})^{n/2}$ .

Remind the line bundle  $\mathcal{A}_R$  on  $\mathcal{L}(R^\perp/R)$  with fibre  $\mathcal{J}_a \otimes \det L_1 \otimes \det(W_a : R)$  at  $L_1$ . Let  $\tilde{\mathcal{L}}(R^\perp/R)$  be the gerb of square roots of  $\mathcal{A}_R$ . So,  $U_R^0 \in \tilde{\mathcal{L}}(R^\perp/R)$ .

Write  $H_R$  for the Heisenberg group  $R^\perp/R \times \mathbb{A}^1$  and  $\mathcal{H}_{U_R}$  for the corresponding category of  $(\bar{U}_R, \chi_{U,R})$ -equivariant perverse sheaves on  $H_R$ . Here  $\bar{U}_R = U_R \times \mathbb{A}^1$  and  $\chi_{U,R}$  is the local system  $\text{pr}^* \mathcal{L}_\psi$  on  $\bar{U}_R$ , where  $\text{pr} : \bar{U}_R \rightarrow \mathbb{A}^1$  is the projection.

Let  $\mathcal{F}_{U_R^0} : \text{D}\mathcal{H}_{U_R} \rightarrow \text{D}(\tilde{\mathcal{L}}(R^\perp/R))$  be the corresponding functor (defined as in [8], Section 3.6). The lattice  $gR \subset W_b(F)$  satisfies the same assumptions, so we have  $U_{gR} = gR_2/gR_1 \in \mathcal{L}(gR^\perp/gR)$ , and  $g(R^\perp) = (gR)^\perp$ . Further,  $U_{gR}^0 = (U_{gR}, \mathcal{B}_1)$  with

$$\mathcal{B}_1 = \mathcal{J}_{U,b} \otimes \det(U_b : gR_1) \otimes \det(\mathcal{O}(-b) : \mathcal{O})^{n/2}$$

equipped with the canonical isomorphism  $\mathcal{B}_1^2 \xrightarrow{\sim} \mathcal{J}_b \otimes \det(U_{gR}) \otimes \det(W_b : gR)$ .

We have  $\tilde{g} = (g, \mathcal{B}_0)$ , where

$$\mathcal{B}_0 = \mathcal{J}_{U,b} \otimes \mathcal{J}_{U,a}^{-1} \otimes \det(U_b : gU_a) \otimes \det(\mathcal{O}(-b) : \mathcal{O}(-a))^{n/2}$$

is equipped with  $\mathcal{B}_0^2 \xrightarrow{\sim} \mathcal{J}_b \otimes \mathcal{J}_a^{-1} \otimes \det(W_b : gW_a)$ . It follows that  $\tilde{g}(U_R^0) \xrightarrow{\sim} U_{gR}^0$  canonically.

Further,  $g$  yields an equivalence  $g : \text{D}\mathcal{H}_{U_R} \xrightarrow{\sim} \text{D}\mathcal{H}_{U_{gR}}$ , and the diagram is canonically 2-commutative

$$\begin{array}{ccc} \text{D}\mathcal{H}_{U_R} & \xrightarrow{\mathcal{F}_{U_R^0}} & \text{D}(\tilde{\mathcal{L}}(R^\perp/R)) \\ \downarrow g & & \downarrow \tilde{g} \\ \text{D}\mathcal{H}_{U_{gR}} & \xrightarrow{\mathcal{F}_{U_{gR}^0}} & \text{D}(\tilde{\mathcal{L}}(gR^\perp/gR)) \end{array} \quad (11)$$

Indeed, this is a consequence of the following isomorphism. We have

$$\tilde{g} : \tilde{\mathcal{L}}(R^\perp/R) \times \tilde{\mathcal{L}}(R^\perp/R) \times H_R \xrightarrow{\sim} \tilde{\mathcal{L}}(gR^\perp/gR) \times \tilde{\mathcal{L}}(gR^\perp/gR) \times H_{gR},$$

and for this isomorphism  $\tilde{g}^* F \xrightarrow{\sim} F$  canonically, where  $F$  on both sides is the corresponding CIO sheaf (introduced in [8], Theorem 1).

**Step 2.** Given c-lattices  $S_1 \subset R_1 \subset R_2 \subset S_2$  in  $U_a(F)$ , similarly define  $S = S_1 \oplus S_2'$  and  $U_S^0 \in \tilde{\mathcal{L}}(S^\perp/S)$  for  $S \subset S^\perp \subset W_a(F)$ . We have a canonical transition functor  $T_{S,R}^U : \text{D}\mathcal{H}_{U_R} \rightarrow \text{D}\mathcal{H}_{U_S}$

defined as in ([8], Section 6.6). Let  $j : \mathcal{L}(S^\perp/S)_R \subset \mathcal{L}(S^\perp/S)$  be the open subscheme of  $L$  satisfying  $L \cap (R/S) = 0$ . We have a projection

$$p_{R/S} : \tilde{\mathcal{L}}(S^\perp/S)_R \rightarrow \tilde{\mathcal{L}}(R^\perp/R)$$

sending  $(L, \mathcal{B}_S)$  to  $(L_R, \mathcal{B}_S)$ , where  $L_R := L \cap R^\perp$ . It is understood that  $\mathcal{B}_S$  is equipped with

$$\mathcal{B}_S^2 \xrightarrow{\sim} \mathcal{J}_a \otimes \det L \otimes \det(W_a : S),$$

and we used the canonical isomorphism  $\det L \otimes \det(W_a : S) \xrightarrow{\sim} \det L_R \otimes \det(W_a : R)$ .

Set  $P_{R/S} = p_{R/S}^*[\dim. \text{rel}(p_{R/S})]$ . Then the following diagram is canonically 2-commutative

$$\begin{array}{ccccc} \mathrm{D}\mathcal{H}_{U_R} & \xrightarrow{\mathcal{F}_{U_R}^0} & \mathrm{D}(\tilde{\mathcal{L}}(R^\perp/R)) & \xrightarrow{P_{R/S}} & \mathrm{D}(\tilde{\mathcal{L}}(S^\perp/S)_R) \\ \downarrow T_{S,R}^U & & & \nearrow j^* & \\ \mathrm{D}\mathcal{H}_{U_S} & \xrightarrow{\mathcal{F}_{U_S}^0} & \mathrm{D}(\tilde{\mathcal{L}}(S^\perp/S)) & & \end{array}$$

Define  ${}_R\mathcal{F}_{U_a(F)}$  as the composition

$$\mathrm{D}\mathcal{H}_{U_R} \xrightarrow{\mathcal{F}_{U_R}^0} \mathrm{D}(\tilde{\mathcal{L}}(R^\perp/R)) \rightarrow \mathrm{D}(\tilde{\mathcal{L}}_d(W_a(F))_R),$$

where the second arrow is the restriction (exact for the perverse t-structures) with respect to the projection  $\tilde{\mathcal{L}}_d(W_a(F))_R \rightarrow \tilde{\mathcal{L}}(R^\perp/R)$ .

The above diagram shows that the following diagram is also 2-commutative

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_{U_R} & \xrightarrow{{}_R\mathcal{F}_{U_a(F)}} & \mathrm{D}(\tilde{\mathcal{L}}_d(W_a(F))_R) \\ \downarrow T_{S,R}^U & & \uparrow j_{S,R}^* \\ \mathrm{D}\mathcal{H}_{U_S} & \xrightarrow{{}_S\mathcal{F}_{U_a(F)}} & \mathrm{D}(\tilde{\mathcal{L}}_d(W_a(F))_S), \end{array}$$

where  $j_{S,R} : \tilde{\mathcal{L}}_d(W_a(F))_R \subset \tilde{\mathcal{L}}_d(W_a(F))_S$  is the natural open immersion.

So, define

$$\mathcal{F}_{U_a(F),R} : \mathrm{D}\mathcal{H}_{U_R} \rightarrow \mathrm{D}(\tilde{\mathcal{L}}_d(W_a(F)))$$

as the functor sending  $K_1$  to the following object  $K_2$ . For c-lattices  $S_1 \subset R_1 \subset R_2 \subset S_2$  as above and  $S = S_1 \oplus S_2'$  declare the restriction of  $K_2$  to  $\tilde{\mathcal{L}}_d(W_a(F))_S$  to be

$$({}_S\mathcal{F}_{U_a(F)} \circ T_{S,R}^U)(K_1)$$

The corresponding projective system (indexed by such  $S$ ) defines an object  $K_2 \in \mathrm{D}(\tilde{\mathcal{L}}_d(W_a(F)))$ .

Further, passing to the limit by  $R$  (of the above form) the functors  $\mathcal{F}_{U_a(F),R}$  yield the desired functor  $\mathcal{F}_{U_a(F)} : \mathrm{D}(U_a^* \otimes \Omega(F)) \rightarrow \mathrm{D}(\tilde{\mathcal{L}}_d(W_a(F)))$ .

Finally, the commutativity of (10) follows from the commutativity of (11).  $\square$

*Remark 4.* We could also argue differently in Proposition 2. For each  $a \in \mathbb{Z}$  and  $L^0 \in \tilde{\mathcal{L}}_d(W_a(F))$  we could first construct an equivalence  $\mathcal{F}_{U_a(F), L^0} : D(U_a^* \otimes \Omega(F)) \xrightarrow{\sim} D\mathcal{H}_L$  as in ([8], Proposition 5) such that for any  $g \in \mathcal{U}_{b,a}$  the diagram is 2-commutative

$$\begin{array}{ccc} D(U_a^* \otimes \Omega(F)) & \xrightarrow{\mathcal{F}_{U_a(F), L^0}} & D\mathcal{H}_L \\ \downarrow g & & \downarrow g \\ D(U_b^* \otimes \Omega(F)) & \xrightarrow{\mathcal{F}_{U_b(F), \tilde{g}(L^0)}} & D\mathcal{H}_{gL} \end{array}$$

with  $\tilde{g} = \rho_{b,a}(g)$ . Here  $\tilde{g}(L^0) \in \tilde{\mathcal{L}}_d(W_b(F))$ . Then we could define  $\mathcal{F}_{U_a(F)}$  as the composition

$$D(U_a^* \otimes \Omega(F)) \xrightarrow{\mathcal{F}_{U_a(F), L^0}} D\mathcal{H}_L \xrightarrow{\mathcal{F}_{L^0}} D(\tilde{\mathcal{L}}_d(W_a(F)))$$

The resulting functor would be (up to a canonical isomorphism) independent of  $L^0 \in \tilde{\mathcal{L}}_d(W_a(F))$ .

#### 4. DUAL PAIR $\mathrm{GSp}_{2n}, \mathrm{GO}_{2m}$

4.1 As in Section 3.2, let  $\mathcal{O}$  be a complete discrete valuation  $k$ -algebra,  $F$  its fraction field,  $\Omega$  the completed module of relative differentials of  $\mathcal{O}$  over  $k$ . For a free  $\mathcal{O}$ -module  $M$  we write  $M_x = M \otimes_{\mathcal{O}} k$  for its geometric fibre.

Fix free  $\mathcal{O}$ -modules  $M_a$  of rank  $2n$ ,  $V_a$  of rank  $2m$ , and  $A_a, C_a$  of rank one with symplectic form  $\wedge^2 M_a \rightarrow A_a$ , a nondegenerate symmetric form  $\mathrm{Sym}^2 V_a \rightarrow C_a$ , and a compatible trivialization  $\det V_a \xrightarrow{\sim} C_a^m$ . Assume also given an isomorphism  $A_a \otimes C_a \xrightarrow{\sim} \Omega(a)$  for each  $a \in \mathbb{Z}$ .

Set  $W_a = M_a \otimes V_a$ , it is equipped with the symplectic form  $\wedge^2 W_a \rightarrow \Omega(a)$ . For  $a \in \mathbb{Z}$  set  $\mathcal{J}_a = C_{a,x}^{-anm}$ , which is of parity zero as  $\mathbb{Z}/2\mathbb{Z}$ -graded. Define  $\tilde{\mathcal{L}}_d(W_a(F))$ ,  $\mathcal{G}_{b,a}$ ,  $G_a$  and  $\tilde{\mathcal{G}}_{b,a}$  as in Section 3.2.

Let  $\mathbb{G} = \mathrm{GSp}_{2n}$  be the symplectic similitude group over  $k$  of semisimple rank  $n$ . Let  $\mathbb{H}$  be the connected component of unity of the split orthogonal similitude group  $\mathrm{GO}_{2m}$  of semisimple rank  $m$ . We may view  $(M_a, A_a)$  (resp.,  $(V_a, C_a)$ ) as a  $\mathbb{G}$ -torsor (resp.,  $\mathbb{H}$ -torsor) on  $\mathrm{Spec} \mathcal{O}$ .

Let  $\mathbb{G}_{b,a}$  be the set of isomorphisms  $M_a(F) \rightarrow M_b(F)$  of  $\mathbb{G}$ -torsors over  $\mathrm{Spec} F$ . Let  $\mathbb{H}_{b,a}$  be the set of isomorphisms  $V_a(F) \rightarrow V_b(F)$  of  $\mathbb{H}$ -torsors over  $\mathrm{Spec} F$ . Let  $\mathcal{T}_{b,a}$  be the set of pairs  $g = (g_1, g_2)$ , where  $g_1 \in \mathbb{G}_{b,a}$ ,  $g_2 \in \mathbb{H}_{b,a}$  such that  $g \in \mathcal{G}_{b,a}$ . That is, the composition

$$\Omega(F) \xrightarrow{\sim} A_a \otimes C_a(F) \xrightarrow{g_1 \otimes g_2} A_b \otimes C_b(F) \xrightarrow{\sim} \Omega(F)$$

must equal to the identity. The natural composition map  $\mathcal{T}_{c,b} \times \mathcal{T}_{b,a} \rightarrow \mathcal{T}_{c,a}$  makes  $\mathcal{T} = \sqcup_{a,b} \mathcal{T}_{b,a}$  into a groupoid.

**Lemma 1.** *Let  $M_i, V$  be a free  $\mathcal{O}_x$ -modules of finite rank, where  $M_2 \subset M_1(F_x)$  is a  $\mathcal{O}_x$ -lattice. Set  $\dim(M_1 : M_2) = \dim(M_1/R) - \dim(M_2/R)$  for any  $\mathcal{O}_x$ -lattice  $R \subset M_1 \cap M_2$ . Then we have a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism<sup>2</sup>*

$$\det(M_1 \otimes V : M_2 \otimes V) \xrightarrow{\sim} \det(M_1 : M_2)^{\mathrm{rk} V} \otimes (\det V_x)^{\dim(M_1 : M_2)} [\dim(M_1 : M_2) \mathrm{rk} V]$$

<sup>2</sup>there may be sign problems, the corresponding isomorphism is well defined at least up to a sign



*Proof* Pick a  $\mathcal{O}_x$ -lattice  $R \subset M_1 \cap M_2$ . It suffices to establish a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\det(M_1 \otimes V : R \otimes V) \xrightarrow{\sim} \det(M_1/R)^{\mathrm{rk} V} \otimes (\det V_x)^{\dim(M_1/R)} [\dim(M_1/R) \mathrm{rk} V]$$

To do so, it suffices to pick a flag  $R = R_0 \subset R_1 \subset \dots \subset R_s = M_1$  of  $\mathcal{O}_x$ -lattices with  $\dim(R_i/R_{i-1}) = 1$ .  $\square$

For  $e \in \mathbb{Z}$  set  $\mathbb{G}_{b,a}^e = \{g \in \mathbb{G}_{b,a} \mid gA_a = A_b(e)\}$  and  $\mathbb{H}_{b,a}^e = \{g \in \mathbb{H}_{b,a} \mid gC_a = C_b(e)\}$ .

Let us construct a canonical section  $\nu_{b,a} : \mathcal{T}_{b,a} \rightarrow \tilde{\mathcal{G}}_{b,a}$  compatible with the groupoids structures. Let  $g = (g_1, g_2) \in \mathcal{T}_{b,a}$  with  $g_1 \in \mathbb{G}_{b,a}^e$ ,  $g_2 \in \mathbb{H}_{b,a}^c$ , so  $e + c = a - b$ . Using Lemma 1 we get a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\begin{aligned} \det(M_b \otimes V_b : (g_1 M_a) \otimes (g_2 V_a)) &\xrightarrow{\sim} \\ \det(M_b : g_1 M_a)^{2m} \otimes \det(V_b : g_2 V_a)^{2n} \otimes (\det V_b)_x^{\dim(M_b : g_1 M_a)} \otimes (\det M_a)_x^{\dim(V_b : g_2 V_a)} &\xrightarrow{\sim} \\ \det(M_b : g_1 M_a)^{2m} \otimes \det(V_b : g_2 V_a)^{2n} \otimes C_{b,x}^{-mne} \otimes A_{a,x}^{-mnc} &\xrightarrow{\sim} \\ \det(M_b : g_1 M_a)^{2m} \otimes \det(V_b : g_2 V_a)^{2n} \otimes C_{b,x}^{-mne} \otimes C_{a,x}^{mnc} \otimes \mathcal{O}((1-a)c)_x^{mn} \end{aligned}$$

We used that  $\dim(M_b : g_1 M_a) = -ne$ ,  $\dim(V_b : g_2 V_a) = -mc$ . Identifying further  $C_a \xrightarrow{g_2} C_b(c)$ , we get

$$\mathcal{J}_b \otimes \mathcal{J}_a^{-1} \otimes \det(W_b : gW_a) \xrightarrow{\sim} \det(M_b : g_1 M_a)^{2m} \otimes \det(V_b : g_2 V_a)^{2n} \otimes C_{b,x}^{2cnm} \otimes \mathcal{O}(c(1+c))_x^{nm}$$

Let  $\nu_{b,a}(g) = (g, \mathcal{B})$ , where

$$\mathcal{B} = \det(M_b : g_1 M_a)^m \otimes \det(V_b : g_2 V_a)^n \otimes C_{b,x}^{cnm} \otimes \mathcal{O}(c(1+c)/2)_x^{nm}$$

is equipped with the induced isomorphism  $\mathcal{B}^2 \xrightarrow{\sim} \mathcal{J}_b \otimes \mathcal{J}_a^{-1} \otimes \det(W_b : gW_a)$ .

We let  $\mathcal{T}$  act on  $\tilde{\mathcal{L}}_d^{ex}$  via  $\nu$ .

4.2 Let  $\mathbb{G}_a = \mathrm{GSp}(M_a)$  and  $\mathbb{H}_a = \mathrm{GO}^0(V_a)$ , the connected component of unity of the group scheme  $\mathrm{GO}(V_a)$  over  $\mathrm{Spec} \mathcal{O}$ . Set

$$\mathcal{T}_a = \{(g_1, g_2) \in (\mathbb{G}_a \times \mathbb{H}_a)(\mathcal{O}) \mid g_1 \otimes g_2 \text{ acts trivially on } A_a \otimes C_a\}$$

The line bundle on  $\mathcal{L}_d(W_a(F))$  with fibre  $\mathcal{J}_a \otimes \det(W_a : L)$  at  $L$  is naturally  $\mathcal{T}_a$ -equivariant (we underline that  $\mathcal{T}_a$  acts nontrivially on  $\mathcal{J}_a$ ). So, it can be seen as a line bundle on the quotient stack  ${}^a\mathcal{X}\mathcal{L} := \mathcal{L}_d(W_a(F))/\mathcal{T}_a$ . We also have the corresponding  $\mu_2$ -gerb

$${}^a\widetilde{\mathcal{X}\mathcal{L}} := \tilde{\mathcal{L}}_d(W_a(F))/\mathcal{T}_a$$

of square roots of this line bundle. The derived category  $D_{\mathcal{T}_a}(\tilde{\mathcal{L}}_d(W_a(F)))$  is defined as in ([7], Section 8.2.2).

The stack  ${}^a\mathcal{X}\mathcal{L}$  classifies: a  $\mathbb{G}$ -torsor  $(M, A)$  over  $\mathrm{Spec}\mathcal{O}$ , a  $\mathbb{H}$ -torsor  $(V, C)$  over  $\mathrm{Spec}\mathcal{O}$  (so, we have a compatible isomorphism  $\det V \xrightarrow{\sim} C^m$ ), an isomorphism  $A \otimes C \xrightarrow{\sim} \Omega(a)$ , and a discrete lagrangian subspace  $L \subset M \otimes V(F)$ .

Let  ${}^a\mathcal{A}_{\mathcal{X}\mathcal{L}}$  be the line bundle over  ${}^a\mathcal{X}\mathcal{L}$  whose fibre at  $(M, A, V, C, L)$  is  $C_x^{-anm} \otimes \det(M \otimes V : L)$ . It is of parity zero as  $\mathbb{Z}/2\mathbb{Z}$ -graded. Then  ${}^a\widetilde{\mathcal{X}\mathcal{L}}$  is the  $\mu_2$ -gerb of square roots of  ${}^a\mathcal{A}_{\mathcal{X}\mathcal{L}}$ .

**4.3.1 HECKE OPERATORS** Denote by  ${}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{X}\mathcal{L}}$  the stack classifying: a point  $(L, M, A, V, C) \in {}^a\mathcal{X}\mathcal{L}$ , a lattice  $M' \subset M(F)$  such that for  $A' = A(a' - a)$  the induced form is regular and nondegenerate  $\wedge^2 M' \rightarrow A'$ . We get a diagram

$${}^a\mathcal{X}\mathcal{L} \xleftarrow{h^\leftarrow} {}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{X}\mathcal{L}} \xrightarrow{h^\rightarrow} {}^{a'}\mathcal{X}\mathcal{L}, \quad (12)$$

where  $h^\leftarrow$  (resp.,  $h^\rightarrow$ ) sends a point of  ${}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{X}\mathcal{L}}$  to  $(L, M, A, V, C)$  (resp., to  $(L, M', A', V, C)$ ).

**Lemma 2.** *For a point  $(L, M, A, M', A', V, C)$  of  ${}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{X}\mathcal{L}}$  there is a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism*

$$C_x^{-a'nm} \otimes \det(M' \otimes V : L) \xrightarrow{\sim} C_x^{-anm} \otimes \det(M \otimes V : L) \otimes \det(M' : M)^{2m} \quad \square$$

Let  ${}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{X}\mathcal{L}} \xrightarrow{\tilde{h}^\rightarrow} {}^{a'}\widetilde{\mathcal{X}\mathcal{L}}$  be map obtained from  $h^\rightarrow$  by the base change  ${}^{a'}\widetilde{\mathcal{X}\mathcal{L}} \rightarrow {}^{a'}\mathcal{X}\mathcal{L}$ . By Lemma 2, we get a diagram

$${}^{a'}\widetilde{\mathcal{X}\mathcal{L}} \xleftarrow{\tilde{h}^\leftarrow} {}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{X}\mathcal{L}} \xrightarrow{\tilde{h}^\rightarrow} {}^{a'}\widetilde{\mathcal{X}\mathcal{L}} \quad (13)$$

Here a point of  ${}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{X}\mathcal{L}}$  is given by a collection  $(L, M, A, M', A', V, C) \in {}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{X}\mathcal{L}}$  together with a one-dimensional space  $\mathcal{B}$  equipped with  $\mathcal{B}^2 \xrightarrow{\sim} C_x^{-a'nm} \otimes \det(M' \otimes V : L)$ . The map  $\tilde{h}^\leftarrow$  sends this point to  $(L, M, A, V, C) \in {}^a\mathcal{X}\mathcal{L}$  together with the one-dimensional space  $\mathcal{B}_1 = \mathcal{B} \otimes \det(M : M')^m$  with the induced isomorphism  $\mathcal{B}_1^2 \xrightarrow{\sim} C_x^{-anm} \otimes \det(M \otimes V : L)$ .

The affine grassmanian  $\mathrm{Gr}_{\mathbb{G}_a} = \mathbb{G}_a(F)/\mathbb{G}_a(\mathcal{O})$  is the ind-scheme classifying  $\mathcal{O}$ -lattices  $R \subset M_a(F)$  such that for some  $r \in \mathbb{Z}$  the induced form  $\wedge^2 R \rightarrow A_a(r)$  is regular and nondegenerate. Write  $\mathrm{Gr}_{\mathbb{G}_a}^r$  for the connected component of  $\mathrm{Gr}_{\mathbb{G}_a}$  given by fixing such  $r$ .

Trivializing a point of  ${}^{a'}\mathcal{X}\mathcal{L}$  (resp., of  ${}^a\mathcal{X}\mathcal{L}$ ) one gets isomorphisms

$$\mathrm{id}^r : {}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{X}\mathcal{L}} \xrightarrow{\sim} (\mathcal{L}_d(W_{a'}(F)) \times \mathrm{Gr}_{\mathbb{G}_{a'}}^{a-a'})/\mathcal{T}_{a'}$$

and

$$\mathrm{id}^l : {}^{a,a'}\mathcal{H}_{\mathbb{G},\mathcal{X}\mathcal{L}} \xrightarrow{\sim} (\mathcal{L}_d(W_a(F)) \times \mathrm{Gr}_{\mathbb{G}_a}^{a'-a})/\mathcal{T}_a,$$

where the corresponding action of  $\mathcal{T}_{a'}$  (resp., of  $\mathcal{T}_a$ ) is diagonal. They lift naturally to a  $\mathcal{T}_{a'}$ -torsor

$$\tilde{\mathcal{L}}_d(W_{a'}(F)) \times \mathrm{Gr}_{\mathbb{G}_{a'}}^{a-a'} \rightarrow {}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{X}\mathcal{L}}$$

and a  $\mathcal{T}_a$ -torsor

$$\tilde{\mathcal{L}}_d(W_a(F)) \times \mathrm{Gr}_{\mathbb{G}_a}^{a'-a} \rightarrow {}^{a,a'}\widetilde{\mathcal{H}}_{\mathbb{G},\mathcal{X}\mathcal{L}}$$

So, for  $K \in D_{\mathcal{T}_a}(\tilde{\mathcal{L}}_d(W_a(F)))$ ,  $K' \in D_{\mathcal{T}_{a'}}(\tilde{\mathcal{L}}_d(W_{a'}(F)))$ ,  $\mathcal{S} \in \text{Sph}_{\mathbb{G}_a}$ ,  $\mathcal{S}' \in \text{Sph}_{\mathbb{G}_{a'}}$  we can form their external products

$$(K \tilde{\boxtimes} \mathcal{S})^l, (K' \tilde{\boxtimes} \mathcal{S}')^r$$

on  ${}^{a,a'}\tilde{\mathcal{H}}_{\mathbb{G},\mathcal{X}\mathcal{L}}$ . The Hecke functor

$$H_{\mathbb{G}}^{\leftarrow} : \text{Sph}_{\mathbb{G}_{a'}} \times D_{\mathcal{T}_{a'}}(\tilde{\mathcal{L}}_d(W_{a'}(F))) \rightarrow D_{\mathcal{T}_a}(\tilde{\mathcal{L}}_d(W_a(F)))$$

is defined by

$$H_{\mathbb{G}}^{\leftarrow}(\mathcal{S}', K') = (\tilde{h}^{\leftarrow})_!(K' \tilde{\boxtimes} * \mathcal{S}')^r$$

It is understood that this informal definition should be made rigorous in a way similar to ([7], Section 4.3).

Write  ${}_b\text{Sph}_{\mathbb{G}_{a'}} \subset \text{Sph}_{\mathbb{G}_{a'}}$  for the full subcategory of objects that vanish off  $\text{Gr}_{\mathbb{G}_{a'}}^b$ . The first argument of  $H_{\mathbb{G}}^{\leftarrow}$  actually lies in  ${}^{a'-a}\text{Sph}_{\mathbb{G}_{a'}}$ .

4.3.2 Let  ${}^{a,a'}\mathcal{H}_{\mathbb{H},\mathcal{X}\mathcal{L}}$  be the stack classifying: a point  $(L, M, A, V, C) \in {}^a\mathcal{X}\mathcal{L}$ , a lattice  $V' \subset V(F)$  such that for  $C' = C(a' - a)$  the induced form  $\text{Sym}^2 V' \rightarrow C'$  is regular and nondegenerate (we also get the isomorphism  $C'^{-m} \otimes \det V' \xrightarrow{\sim} C'^{-m} \otimes \det V \xrightarrow{\sim} \mathcal{O}$ ). As for  $\mathbb{G}$ , we get a diagram

$$\begin{array}{ccccc} {}^a\widetilde{\mathcal{X}\mathcal{L}} & \xleftarrow{\tilde{h}^{\leftarrow}} & {}^{a,a'}\tilde{\mathcal{H}}_{\mathbb{H},\mathcal{X}\mathcal{L}} & \xrightarrow{\tilde{h}^{\rightarrow}} & {}^{a'}\widetilde{\mathcal{X}\mathcal{L}} \\ \downarrow & & \downarrow & & \downarrow \\ {}^a\mathcal{X}\mathcal{L} & \xleftarrow{h^{\leftarrow}} & {}^{a,a'}\mathcal{H}_{\mathbb{H},\mathcal{X}\mathcal{L}} & \xrightarrow{h^{\rightarrow}} & {}^{a'}\mathcal{X}\mathcal{L}, \end{array} \quad (14)$$

where  $h^{\leftarrow}$  (resp.  $h^{\rightarrow}$ ) sends  $(L, M, A, V, C, V', C')$  to  $(L, M, A, V, C)$  (resp., to  $(L, M, A, V', C')$ ), the vertical arrows are  $\mu_2$ -gerbs, and the right square is cartesian (thus defining the stack  ${}^{a,a'}\tilde{\mathcal{H}}_{\mathbb{H},\mathcal{X}\mathcal{L}}$ ).

A point of  ${}^{a,a'}\tilde{\mathcal{H}}_{\mathbb{H},\mathcal{X}\mathcal{L}}$  is given by  $(L, M, A, V, C, V', C') \in {}^{a,a'}\mathcal{H}_{\mathbb{H},\mathcal{X}\mathcal{L}}$  and a one-dimensional space  $\mathcal{B}$  equipped with

$$\mathcal{B}^2 \xrightarrow{\sim} (C'_x)^{-a'nm} \otimes \det(M \otimes V' : L)$$

The map  $\tilde{h}^{\leftarrow}$  sends this point to  $(L, M, A, V, C) \in {}^a\mathcal{X}\mathcal{L}$ , the one-dimensional space  $\mathcal{B}_1$  with  $\mathcal{B}_1^2 \xrightarrow{\sim} C_x^{-anm} \otimes \det(M \otimes V : L)$ , where

$$\mathcal{B}_1 = \mathcal{B} \otimes C_x^{nm(a'-a)} \otimes \det(V : V')^n \otimes \mathcal{O}(\frac{1}{2}nm(a-a')(a-a'-1))_x$$

The affine grassmanian  $\text{Gr}_{\mathbb{H}_a}$  classifies lattices  $V' \subset V_a(F)$  such that the induced symmetric form  $\text{Sym}^2 V' \rightarrow C_a(b)$  is regular and nondegenerate for some  $b \in \mathbb{Z}$ . Write  $\text{Gr}_{\mathbb{H}_a}^b$  for the locus of  $\text{Gr}_{\mathbb{H}_a}$  given by fixing this  $b$ . For  $m \geq 2$  there is an exact sequence  $0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1(\mathbb{H}_a) \rightarrow \mathbb{Z} \rightarrow 0$ , so if  $m \geq 2$  then  $\text{Gr}_{\mathbb{H}_a}^b$  is a union of two connected components of  $\text{Gr}_{\mathbb{H}_a}$ . Write  ${}_b\text{Sph}_{\mathbb{H}_a} \subset \text{Sph}_{\mathbb{H}_a}$  for the full subcategory of objects that vanish off  $\text{Gr}_{\mathbb{H}_a}^b$ .

The Hecke functor

$$H_{\mathbb{H}}^{\leftarrow} : \text{Sph}_{\mathbb{H}_{a'}} \times D_{\mathcal{T}_{a'}}(\tilde{\mathcal{L}}_d(W_{a'}(F))) \rightarrow D_{\mathcal{T}_a}(\tilde{\mathcal{L}}_d(W_a(F)))$$

is defined as in Section 4.3.1 using the diagram (14).

For each  $a \in \mathbb{Z}$  a trivialization  $\alpha$  of the  $\mathbb{G}$ -torsor  $(M_a, A_a)$  on  $\text{Spec } \mathcal{O}$  yields an isomorphism  $\bar{\alpha} : \text{Gr}_{\mathbb{G}_a} \xrightarrow{\sim} \text{Gr}_{\mathbb{G}}$ . The induced equivalences  $\bar{\alpha}^* : \text{Sph}_{\mathbb{G}} \xrightarrow{\sim} \text{Sph}_{\mathbb{G}_a}$  are canonically 2-isomorphic for different  $\alpha$ 's. In what follows we sometimes identify these two categories in this way. Similarly, we identify  $\text{Sph}_{\mathbb{H}_a} \xrightarrow{\sim} \text{Sph}_{\mathbb{H}}$ .

4.4 Let  $S_{W_0(F)} \in \mathcal{P}_{\mathcal{T}_0}(\tilde{\mathcal{L}}_d(W_0(F)))$  be the theta-sheaf introduced in ([8], Section 6.5). This is a  $\mathcal{T}_0$ -equivariant object of the Weil category  $W(\tilde{\mathcal{L}}_d(W_0(F)))$ . Here is the main result of Section 4.

**Theorem 3.** 1) Assume  $m \leq n$ . There is a homomorphism  $\kappa : \check{\mathbb{H}} \times \mathbb{G}_m \rightarrow \check{\mathbb{G}}$  such that for the corresponding geometric restriction functor  $\text{gRes}^\kappa : \text{Sph}_{\mathbb{G}} \rightarrow \text{D Sph}_{\mathbb{H}}$  we have an isomorphism in  $\text{D}_{\mathcal{T}_a}(\tilde{\mathcal{L}}_d(W_a(F)))$

$$\text{H}_{\mathbb{G}}^{\leftarrow}(\mathcal{S}, S_{W_0(F)}) \xrightarrow{\sim} \text{H}_{\mathbb{H}}^{\leftarrow}(*\text{gRes}^\kappa(\mathcal{S}), S_{W_0(F)})$$

functorial in  $\mathcal{S} \in {}_{-a}\text{Sph}_{\mathbb{G}}$ .

2) Assume  $m > n$ . There is a homomorphism  $\kappa : \check{\mathbb{G}} \times \mathbb{G}_m \rightarrow \check{\mathbb{H}}$  such that for the corresponding geometric restriction functor  $\text{gRes}^\kappa : \text{Sph}_{\mathbb{H}} \rightarrow \text{D Sph}_{\mathbb{G}}$  we have an isomorphism in  $\text{D}_{\mathcal{T}_a}(\tilde{\mathcal{L}}_a(W_a(F)))$

$$\text{H}_{\mathbb{H}}^{\leftarrow}(\mathcal{S}, S_{W_0(F)}) \xrightarrow{\sim} \text{H}_{\mathbb{G}}^{\leftarrow}(\text{gRes}^\kappa(*\mathcal{S}), S_{W_0(F)})$$

functorial in  $\mathcal{S} \in {}_{-a}\text{Sph}_{\mathbb{H}}$ .

The proof occupies the rest of Section 4. The explicit formulas for  $\kappa$  are found in Section 4.8.9.

4.5 Assume given a decomposition  $M_a = L_a \oplus (L_a^* \otimes A_a)$ , where  $L_a$  is a free  $\mathcal{O}$ -module of rank  $n$ ,  $L_a$  and  $L_a^* \otimes A_a$  are lagrangians, and the form is given by the canonical pairing between  $L_a$  and  $L_a^*$ . Assume given a similar decomposition  $V_a = U_a \oplus (U_a^* \otimes C_a)$  for  $V_a$ , here  $U_a$  is a free  $\mathcal{O}$ -module of rank  $m$ .

Write  $Q(\mathbb{G}_a) \subset \mathbb{G}_a$  and  $Q(\mathbb{H}_a)$  for the Levi subgroups preserving the above decompositions. Set

$$Q\mathbb{G}\mathbb{H}_a = \{g = (g_1, g_2) \in Q(\mathbb{G}_a) \times Q(\mathbb{H}_a) \mid g \in \mathcal{T}_a\}$$

$$\mathbb{G}Q\mathbb{H}_a = \{g = (g_1, g_2) \in \mathbb{G}_a \times Q(\mathbb{H}_a) \mid g \in \mathcal{T}_a\}$$

$$\mathbb{H}Q\mathbb{G}_a = \{g = (g_1, g_2) \in \mathbb{H}_a \times Q(\mathbb{G}_a) \mid g \in \mathcal{T}_a\}$$

We view all of them as group schemes over  $\text{Spec } \mathcal{O}$ . We also pick Levi subgroups  $Q(\mathbb{G}) \subset \mathbb{G}$  and  $Q(\mathbb{H}) \subset \mathbb{H}$  which identify with the above over  $\text{Spec } \mathcal{O}$ .

The affine grassmanian  $\text{Gr}_{Q(\mathbb{G}_a)}$  classifies pairs of lattices  $L' \subset L_a(F)$ ,  $A' \subset A_a(F)$ . For  $b \in \mathbb{Z}$  write  $\text{Gr}_{Q(\mathbb{G}_a)}^b$  for the locus of  $\text{Gr}_{Q(\mathbb{G}_a)}$  given by  $A' = A_a(b)$ . Write  ${}_b\text{Sph}_{Q(\mathbb{G}_a)} \subset \text{Sph}_{Q(\mathbb{G}_a)}$  for the full subcategory of objects that vanish off  $\text{Gr}_{Q(\mathbb{G}_a)}^b$ . As in Section 4.4, we identify canonically  $\text{Sph}_{Q(\mathbb{G})} \xrightarrow{\sim} \text{Sph}_{Q(\mathbb{G}_a)}$ . The geometric restriction  $\text{gRes} : \text{Sph}_{\mathbb{G}} \rightarrow \text{Sph}_{Q(\mathbb{G})}$  corresponding to the inclusion of the Langlands dual groups  $\check{Q}(\mathbb{G}) \hookrightarrow \check{\mathbb{G}}$  yields a faithful functor  ${}_b\text{Sph}_{\mathbb{G}} \rightarrow {}_b\text{Sph}_{Q(\mathbb{G})}$  for each  $b$ . And similarly for  $\mathbb{H}$ .

For  $b, a \in \mathbb{Z}$  write  $Q(\mathbb{G}_{b,a})$  for the set of isomorphisms  $(L_a(F) \rightarrow L_b(F), A_a(F) \rightarrow A_b(F))$  of  $\mathrm{GL}_n \times \mathbb{G}_m$ -torsors over  $\mathrm{Spec} F$ . Let  $Q(\mathbb{H}_{b,a})$  be the set of isomorphisms  $(U_a(F) \rightarrow U_b(F), C_a(F) \rightarrow C_b(F))$  of  $\mathrm{GL}_m \times \mathbb{G}_m$ -torsors over  $\mathrm{Spec} F$ . Set

$$Q\mathbb{G}\mathbb{H}_{b,a} = \{g = (g_1, g_2) \in Q(\mathbb{G}_{b,a}) \times Q(\mathbb{H}_{b,a}) \mid g \in \mathcal{G}_{b,a}\}$$

$$\mathbb{G}Q\mathbb{H}_{b,a} = \{g = (g_1, g_2) \in \mathbb{G}_{b,a} \times Q(\mathbb{H}_{b,a}) \mid g \in \mathcal{G}_{b,a}\}$$

$$\mathbb{H}Q\mathbb{G}_{b,a} = \{g = (g_1, g_2) \in \mathbb{H}_{b,a} \times Q(\mathbb{G}_{b,a}) \mid g \in \mathcal{G}_{b,a}\}$$

Set  $\Upsilon_a = L_a^* \otimes A_a \otimes V_a$  and  $\Pi_a = U_a^* \otimes C_a \otimes M_a$ . For  $a \in \mathbb{Z}$  and any  $L^0 \in \tilde{\mathcal{L}}_d(W_a(F))$  we have the equivalences

$$\mathcal{F}_{L_a \otimes V_a(F), L^0} : D(\Upsilon_a(F)) \xrightarrow{\sim} D\mathcal{H}_L$$

and  $\mathcal{F}_{U_a \otimes M_a(F), L^0} : D(\Pi_a(F)) \xrightarrow{\sim} D\mathcal{H}_L$  defined as in Remark 4.

Remind that for free  $\mathcal{O}$ -modules of finite type  $\mathcal{V}, \mathcal{U}$  one has the partial Fourier transform

$$\mathrm{Four}_\psi : D(\mathcal{V}(F) \oplus \mathcal{U}(F)) \xrightarrow{\sim} D(\mathcal{V}^* \otimes \Omega(F) \oplus \mathcal{U}(F))$$

normalized to preserve perversity and purity (cf. [7], Section 4.8 for the definition). Thus, the decompositions

$$\Pi_a \xrightarrow{\sim} U_a^* \otimes C_a \otimes L_a \oplus U_a^* \otimes L_a^* \otimes \Omega(a)$$

and

$$\Upsilon_a \xrightarrow{\sim} L_a^* \otimes A_a \otimes U_a \oplus U_a^* \otimes L_a^* \otimes \Omega(a)$$

yield the partial Fourier transform, which we denote

$$\zeta_a : D(\Upsilon_a(F)) \xrightarrow{\sim} D(\Pi_a(F))$$

One checks that  $\zeta_a$  is canonically isomorphic to the functor  $\mathcal{F}_{U_a \otimes M_a(F), L^0}^{-1} \circ \mathcal{F}_{L_a \otimes V_a(F), L^0}$  for any  $L^0 \in \tilde{\mathcal{L}}_d(W_a(F))$ .

4.6.1 It is convenient to denote  $\tilde{\Upsilon}_a = L_a \otimes V_a$  and  $\bar{\Pi}_a = U_a \otimes M_a$ . For the decomposition  $W_a = \bar{\Pi}_a \oplus \bar{\Pi}_a^* \otimes \Omega(a)$  we define a  $\mathbb{Z}/2\mathbb{Z}$ -graded line (purely of parity zero)

$$\mathcal{J}_{\bar{\Pi},a} = \mathcal{O}((1-a)a/2)_x^{nm} \otimes (\det U_{a,x})^{-na}$$

equipped with a natural  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\mathcal{J}_{\bar{\Pi},a}^2 \xrightarrow{\sim} \mathcal{J}_a \otimes (\det \bar{\Pi}_a)_x^{-a}$$

It yields a section  ${}_{\bar{\Pi}}\rho_{b,a} : \mathbb{G}Q\mathbb{H}_{b,a} \rightarrow \tilde{\mathcal{G}}_{b,a}$  defined as in Section 3.3.

For the decomposition

$$W_a = \tilde{\Upsilon}_a \oplus \tilde{\Upsilon}_a^* \otimes \Omega(a)$$

define a  $\mathbb{Z}/2\mathbb{Z}$ -graded line (purely of parity zero)

$$\mathcal{J}_{\tilde{\Upsilon},a} = C_{a,x}^{-mna} \otimes (\det L_{a,x})^{-ma}$$

equipped with a natural  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\mathcal{J}_{\tilde{\Upsilon},a}^2 \xrightarrow{\sim} \mathcal{J}_a \otimes (\det \tilde{\Upsilon}_{a,x})^{-a}$$

It yields a section  $\tilde{\Upsilon}\rho_{b,a} : \mathbb{H}Q\mathbb{G}_{b,a} \rightarrow \tilde{\mathcal{G}}_{b,a}$  defined as in Section 3.3.

From definitions one derives the following.

**Lemma 3.** *For  $a, b \in \mathbb{Z}$  the following diagrams are canonically 2-commutative*

$$\begin{array}{ccc} \mathcal{T}_{b,a} & \xrightarrow{\nu_{b,a}} & \tilde{\mathcal{G}}_{b,a} \\ \uparrow & \nearrow_{\Pi\rho_{b,a}} & \\ \mathbb{G}Q\mathbb{H}_{b,a} & & \end{array} \quad \begin{array}{ccc} \mathcal{T}_{b,a} & \xrightarrow{\nu_{b,a}} & \tilde{\mathcal{G}}_{b,a} \\ \uparrow & \nearrow_{\tilde{\Upsilon}\rho_{b,a}} & \\ \mathbb{H}Q\mathbb{G}_{b,a} & & \end{array} \quad \square$$

For  $a \in \mathbb{Z}$  we have the functors  $\mathcal{F}_{\tilde{\Upsilon}_a(F)} : D(\Upsilon_a(F)) \rightarrow D(\tilde{\mathcal{L}}_d(W_a(F)))$  and  $\mathcal{F}_{\tilde{\Pi}_a(F)} : D(\Pi_a(F)) \rightarrow D(\tilde{\mathcal{L}}_d(W_a(F)))$  defined in Proposition 2. Note that the diagram is canonically 2-commutative

$$\begin{array}{ccc} D(\Upsilon_a(F)) & \xrightarrow{\mathcal{F}_{\tilde{\Upsilon}_a(F)}} & D(\tilde{\mathcal{L}}_d(W_a(F))) \\ \downarrow \zeta_a & \nearrow_{\mathcal{F}_{\tilde{\Pi}_a(F)}} & \\ D(\Pi_a(F)) & & \end{array}$$

*Remark 5.* The following structure emerge. For each  $g \in \mathcal{T}_{b,a}$  we get functors that fit into a 2-commutative diagram

$$\begin{array}{ccc} D(\Pi_a(F)) & \xrightarrow{g} & D(\Pi_b(F)) \\ \uparrow \zeta_a & & \uparrow \zeta_b \\ D(\Upsilon_a(F)) & \xrightarrow{g} & D(\Upsilon_b(F)) \end{array}$$

They are compatible with the groupoid structure on  $\mathcal{T}$ . Indeed, one first defines these functors separately for  $\mathbb{G}Q\mathbb{H}_{b,a} \subset \mathcal{T}_{b,a}$  and for  $\mathbb{H}Q\mathbb{G}_{b,a} \subset \mathcal{T}_{b,a}$  using the models  $\Pi$  and  $\Upsilon$  respectively. This is sufficient because any  $g \in \mathcal{T}_{b,a}$  writes as a composition  $g = g'' \circ g'$  with  $g'' \in \mathbb{H}Q\mathbb{G}_{b,b}$  and  $g' \in \mathbb{G}Q\mathbb{H}_{b,a}$ . The arrows in the above diagram are equivalences.

4.7 We have the full subcategories (stable under subquotients)

$$P_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)) \subset P_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Upsilon_a(F)) \subset P(\Upsilon_a(F))$$

$$P_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \subset P_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \subset P(\Pi_a(F)),$$

and  $\zeta_a$  yields an equivalence  $\zeta_a : P_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Upsilon_a(F)) \xrightarrow{\sim} P_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Pi_a(F))$ .

**Definition 2.** For  $a \in \mathbb{Z}$  let  $\text{Weil}_a$  be the category of triples  $(\mathcal{F}_1, \mathcal{F}_2, \beta)$ , where

$$\mathcal{F}_1 \in P_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)), \quad \mathcal{F}_2 \in P_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)),$$

and  $\beta : \zeta_a(\mathcal{F}_1) \xrightarrow{\sim} \mathcal{F}_2$  is an isomorphism in  $P_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Pi_a(F))$ . Write  $D\text{Weil}_a$  for the category obtained by replacing everywhere in the above definition  $P$  by  $DP$ .

Clearly,  $\mathcal{W}eil_a$  is an abelian category, and the forgetful functors  $f_{\mathbb{H}} : \mathcal{W}eil_a \rightarrow \mathcal{P}_{\mathbb{H}Q\mathbb{H}_a(\mathcal{O})}(\Upsilon_a(F))$  and  $f_{\mathbb{G}} : \mathcal{W}eil_a \rightarrow \mathcal{P}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F))$  are full embeddings. By Proposition 2, we get a functor

$$\mathcal{F}_{\mathcal{W}eil_a} : \mathcal{W}eil_a \rightarrow \mathcal{P}_{\mathcal{T}_a}(\tilde{\mathcal{L}}_d(W_a(F)))$$

sending  $(\mathcal{F}_1, \mathcal{F}_2, \beta)$  to  $\mathcal{F}_{\Upsilon_a(F)}(\mathcal{F}_1)$ .

Let  $I_0 \in \mathcal{P}_{\mathbb{H}Q\mathbb{G}_0(\mathcal{O})}(\Upsilon_0(F))$  denote the constant perverse sheaf on  $\Upsilon_0$  extended by zero to  $\Upsilon_0(F)$ . Remind that  $\zeta_0(I_0)$  is the constant perverse sheaf on  $\Pi_0$  extended by zero to  $\Pi_0(F)$ . The object  $\zeta_0(I_0)$  will also be denoted  $I_0$  by abuse of notation. So,  $I_0 \in \mathcal{W}eil_0$  naturally.

By definition of the theta-sheaf, we have canonically  $\mathcal{F}_{\mathcal{W}eil_0}(I_0) \xrightarrow{\sim} S_{W_0(F)}$  in  $\mathcal{P}_{\mathcal{T}_0}(\tilde{\mathcal{L}}_d(W_0(F)))$ .

#### 4.8 MORE HECKE OPERATORS

4.8.1 For  $a \in \mathbb{Z}$  let  ${}^a\mathcal{X}\Pi$  be the stack classifying: a  $\mathrm{GL}_m \times \mathbb{G}_m$ -torsor  $(U, C)$  over  $\mathrm{Spec} \mathcal{O}$ ,  $\mathbb{G}$ -torsor  $(M, A, \wedge^2 M \rightarrow A)$  over  $\mathrm{Spec} \mathcal{O}$ , an isomorphism  $A \otimes C \xrightarrow{\sim} \Omega(a)$ , and a section  $s \in U^* \otimes M^* \otimes \Omega(F)$ .

Informally, we may view  $\mathcal{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F))$  as the derived category on  ${}^a\mathcal{X}\Pi$ . For  $a, a' \in \mathbb{Z}$  we are going to define a Hecke functor

$$H_{\mathbb{G}}^{\leftarrow} : {}_{a'-a}\mathrm{Sph}_{\mathbb{G}} \times \mathcal{D}_{\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})}(\Pi_{a'}(F)) \rightarrow \mathcal{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \quad (15)$$

To do so, consider the stack  ${}^{a,a'}\mathcal{H}_{\mathcal{X}\Pi}$  classifying: a point of  ${}^a\mathcal{X}\Pi$  as above, a lattice  $M' \subset M(F)$  such that for  $A' = A(a' - a)$  the induced form  $\wedge^2 M' \rightarrow A'$  is regular and nondegenerate.

We get a diagram

$${}^a\mathcal{X}\Pi \xleftarrow{h^{\leftarrow}} {}^{a,a'}\mathcal{H}_{\mathcal{X}\Pi} \xrightarrow{h^{\rightarrow}} {}^{a'}\mathcal{X}\Pi,$$

where  $h^{\leftarrow}$  sends the above collection to  $(U, C, M, A, s)$ , the map  $h^{\rightarrow}$  sends the above collection to  $(U, C, M', A', s')$ , where  $s'$  is the image of  $s$  under  $U^* \otimes M^* \otimes \Omega(F) \xrightarrow{\sim} U^* \otimes M'^* \otimes \Omega(F)$ .

Trivializing a point of  ${}^{a'}\mathcal{X}\Pi$  (resp., of  ${}^a\mathcal{X}\Pi$ ), one gets isomorphisms

$$\mathrm{id}^r : {}^{a,a'}\mathcal{H}_{\mathcal{X}\Pi} \xrightarrow{\sim} (\Pi_{a'}(F) \times \mathrm{Gr}_{\mathbb{G}_{a'}}^{a-a'}) / \mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})$$

and

$$\mathrm{id}^l : {}^{a,a'}\mathcal{H}_{\mathcal{X}\Pi} \xrightarrow{\sim} (\Pi_a(F) \times \mathrm{Gr}_{\mathbb{G}_a}^{a'-a}) / \mathbb{G}Q\mathbb{H}_a(\mathcal{O})$$

So for

$$K \in \mathcal{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)), \quad K' \in \mathcal{D}_{\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})}(\Pi_{a'}(F)), \quad \mathcal{S} \in {}_{a'-a}\mathrm{Sph}_{\mathbb{G}}, \quad \mathcal{S}' \in {}_{a-a'}\mathrm{Sph}_{\mathbb{G}}$$

one can form the twisted exterior products  $(K \tilde{\boxtimes} \mathcal{S})^l$  and  $(K' \tilde{\boxtimes} \mathcal{S}')^r$  on  ${}^{a,a'}\mathcal{H}_{\mathcal{X}\Pi}$ . The functor (15) is defined by

$$H_{\mathbb{G}}^{\leftarrow}(\mathcal{S}', K') \xrightarrow{\sim} h_{\dagger}^{\leftarrow}(K' \tilde{\boxtimes} \mathcal{S}')^r$$

It is understood that this informal definition should be made rigorous in the same way as in ([7], Section 4.3).

4.8.2 For  $a \in \mathbb{Z}$  let  ${}^a\mathcal{X}\Upsilon$  be the stack classifying: a  $\mathrm{GL}_n \times \mathbb{G}_m$ -torsor  $(L, A)$  over  $\mathrm{Spec} \mathcal{O}$ , a  $\mathbb{H}$ -torsor  $(V, C)$  over  $\mathcal{O}$  (so, we are also given a compatible trivialization  $\det V \xrightarrow{\sim} C^m$ ), an isomorphism  $A \otimes C \xrightarrow{\sim} \Omega(a)$ , and a section  $s \in L^* \otimes V^* \otimes \Omega(F)$ .

We may view  $\mathrm{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))$  as the derived category on  ${}^a\mathcal{X}\Upsilon$ . For  $a, a' \in \mathbb{Z}$  we define a Hecke functor

$$\mathrm{H}_{\mathbb{H}}^{\leftarrow} : {}_{a'-a}\mathrm{Sph}_{\mathbb{H}} \times \mathrm{D}_{\mathbb{H}Q\mathbb{G}_{a'}(\mathcal{O})}(\Upsilon_{a'}(F)) \rightarrow \mathrm{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)) \quad (16)$$

as follows. Let  ${}^{a,a'}\mathcal{H}_{\mathcal{X}\Upsilon}$  be the stack classifying: a point of  ${}^a\mathcal{X}\Upsilon$  as above, a lattice  $V' \subset V(F)$  such that for  $C' = C(a' - a)$  the induced form  $\mathrm{Sym}^2 V' \rightarrow C'$  is regular and nondegenerate (we also get a compatible trivialization

$$C'^{-m} \otimes \det V' \xrightarrow{\sim} C^{-m} \otimes \det V \xrightarrow{\sim} \mathcal{O},$$

so  $(V', C')$  is a  $\mathbb{H}$ -torsor over  $\mathrm{Spec} \mathcal{O}$ ).

As in Section 4.8.1, we get a diagram

$${}^a\mathcal{X}\Upsilon \xleftarrow{h^{\leftarrow}} {}^{a,a'}\mathcal{H}_{\mathcal{X}\Upsilon} \xrightarrow{h^{\rightarrow}} {}^{a'}\mathcal{X}\Upsilon$$

and the desired functor (16).

4.8.3 We need the following lemma. Write  ${}^a\mathcal{X}\check{\Pi}$  for the stack classifying: a  $\mathrm{GL}_m \times \mathbb{G}_m$ -torsor  $(U, C)$  over  $\mathrm{Spec} \mathcal{O}$ , a  $\mathbb{G}$ -torsor  $(M, A, \wedge^2 M \rightarrow A)$  over  $\mathrm{Spec} \mathcal{O}$ , an isomorphism  $A \otimes C \xrightarrow{\sim} \Omega(a)$ , and a section  $s_1 \in U \otimes M(F)$ . View  $\mathrm{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a^* \otimes \Omega(F))$  as the derived category on  ${}^a\mathcal{X}\check{\Pi}$ .

For  $a, a' \in \mathbb{Z}$  define the Hecke functor

$$\mathrm{H}_{\mathbb{G}}^{\leftarrow} : {}_{a'-a}\mathrm{Sph}_{\mathbb{G}} \times \mathrm{D}_{\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})}(\Pi_{a'}^* \otimes \Omega(F)) \rightarrow \mathrm{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a^* \otimes \Omega(F)) \quad (17)$$

as follows. Let  ${}^{a,a'}\mathcal{H}_{\mathcal{X}\check{\Pi}}$  be the stack classifying: a point of  ${}^a\mathcal{X}\check{\Pi}$  as above, a lattice  $M' \subset M(F)$  such that for  $A' = A(a' - a)$  the induced form  $\wedge^2 M' \rightarrow A'$  is regular and nondegenerate.

As above, we get a diagram

$${}^a\mathcal{X}\check{\Pi} \xleftarrow{h^{\leftarrow}} {}^{a,a'}\mathcal{H}_{\mathcal{X}\check{\Pi}} \xrightarrow{h^{\rightarrow}} {}^{a'}\mathcal{X}\check{\Pi},$$

where  $h^{\leftarrow}$  sends the above point to  $(U, C, M, A, s_1)$ , the map  $h^{\rightarrow}$  sends the above point to  $(U, C, M', A', s'_1)$ , where  $s'_1$  is the image of  $s_1$  under  $U \otimes M(F) \xrightarrow{\sim} U \otimes M'(F)$ . Now (17) is defined in a way similar to (15).

Write  $\mathrm{Four}_{\psi} : \mathrm{D}_{\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})}(\Pi_{a'}^* \otimes \Omega(F)) \xrightarrow{\sim} \mathrm{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a^* \otimes \Omega(F))$  for the Fourier transform (normalized as in Section 2.1). The following is standard (cf. also [7], Lemma 11).

**Lemma 4.** *We have a canonical isomorphism in  $\mathrm{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a^* \otimes \Omega(F))$*

$$\mathrm{H}_{\mathbb{G}}^{\leftarrow}(\mathcal{S}, \mathrm{Four}_{\psi}(K)) \xrightarrow{\sim} \mathrm{Four}_{\psi} \mathrm{H}_{\mathbb{G}}^{\leftarrow}(\mathcal{S}, K)$$

*functorial in  $\mathcal{S} \in {}_{a'-a}\mathrm{Sph}_{\mathbb{G}}$ ,  $K \in \mathrm{D}_{\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})}(\Pi_{a'}(F))$ .  $\square$*



4.8.4 Write  $P_{\mathbb{H}_a} \subset \mathbb{H}_a$  (resp.,  $P_{\mathbb{H}_a}^- \subset \mathbb{H}_a$ ) for the parabolic subgroup preserving  $U_a$  (resp.,  $U_a^* \otimes C_a$ ). Let  $U_{\mathbb{H}_a} \subset P_{\mathbb{H}_a}$  and  $U_{\mathbb{H}_a}^- \subset P_{\mathbb{H}_a}^-$  denote their unipotent radicals. We view all of them as group schemes over  $\text{Spec } \mathcal{O}$ . Then  $U_{\mathbb{H}_a} \xrightarrow{\sim} C_a^* \otimes \wedge^2 U_a$  and  $U_{\mathbb{H}_a}^- \xrightarrow{\sim} C_a \otimes \wedge^2 U_a^*$  canonically.

Similarly, let  $P_{\mathbb{G}_a} \subset \mathbb{G}_a$  (resp.,  $P_{\mathbb{G}_a}^- \subset \mathbb{G}_a$ ) be the parabolic subgroup preserving  $L_a$  (resp.,  $L_a^* \otimes A_a$ ). Write  $U_{\mathbb{G}_a} \subset P_{\mathbb{G}_a}$  and  $U_{\mathbb{G}_a}^- \subset P_{\mathbb{G}_a}^-$  for their unipotent radicals. All of them are group schemes over  $\text{Spec } \mathcal{O}$ . We have canonically

$$U_{\mathbb{G}_a} \xrightarrow{\sim} A_a^* \otimes \text{Sym}^2 L_a, \quad U_{\mathbb{G}_a}^- \xrightarrow{\sim} A_a \otimes \text{Sym}^2 L_a^*$$

View  $v \in \Pi_a(F)$  as a map  $v : C_a^* \otimes U_a(F) \rightarrow M_a(F)$ . For  $v \in \Pi_a(F)$  let  $s_{\Pi}(v)$  denote the composition

$$\wedge^2(U_a \otimes C_a^{-1})(F) \xrightarrow{\wedge^2 v} \wedge^2 M_a(F) \rightarrow A_a(F)$$

Let  $\text{Char}(\Pi_a) \subset \Pi_a(F)$  denote the ind-subscheme of  $v \in \Pi_a(F)$  such that  $s_{\Pi}(v) : C_a^* \otimes \wedge^2 U_a \rightarrow \Omega$  is regular. An object  $K \in \text{P}(\Upsilon_a(F))$  is  $U_{\mathbb{H}_a}(\mathcal{O})$ -equivariant iff  $\zeta_a(K)$  is the extension by zero from  $\text{Char}(\Pi_a)$ .

View  $v \in \Upsilon_a(F)$  as a map  $v : L_a \otimes A_a^*(F) \rightarrow V_a(F)$ . For  $v \in \Upsilon_a(F)$  let  $s_{\Upsilon}(v)$  denote the composition

$$\text{Sym}^2(A_a^* \otimes L_a) \xrightarrow{\text{Sym}^2 v} \text{Sym}^2 V_a(F) \rightarrow C_a(F)$$

Write  $\text{Char}(\Upsilon_a) \subset \Upsilon_a(F)$  for the ind-subscheme of  $v \in \Upsilon_a(F)$  such that  $s_{\Upsilon}(v) : A_a^* \otimes \text{Sym}^2 L_a \rightarrow \Omega$  is regular. An object  $K \in \text{P}(\Pi_a(F))$  is  $U_{\mathbb{G}_a}(\mathcal{O})$ -equivariant iff  $\zeta_a^{-1}(K)$  is the extension by zero from  $\text{Char}(\Upsilon_a)$ .

The next result follows from ([7], Lemma 13).

**Lemma 5.** *The full subcategory  $\text{P}_{\mathbb{H}\mathbb{Q}\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)) \subset \text{P}(\Upsilon_a(F))$  is the intersection of the full subcategories*

$$\text{P}_{U_{\mathbb{H}_a}(\mathcal{O})}(\Upsilon_a(F)) \cap \text{P}_{U_{\mathbb{H}_a}^-(\mathcal{O})}(\Upsilon_a(F)) \cap \text{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Upsilon_a(F))$$

inside  $\text{P}(\Upsilon_a(F))$ .  $\square$

**Proposition 3.** *For  $a \in \mathbb{Z}$  the functor  $-_a \text{Sph}_{\mathbb{G}} \rightarrow \text{D}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a(F))$  sending  $\mathcal{S}$  to  $\text{H}_{\mathbb{G}}^{\leftarrow}(\mathcal{S}, I_0)$  factors naturally into*

$$-_a \text{Sph}_{\mathbb{G}} \rightarrow \text{DWeil}_a \rightarrow \text{D}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a(F))$$

*For  $a \in \mathbb{Z}$  the functor  $-_a \text{Sph}_{\mathbb{H}} \rightarrow \text{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))$  sending  $\mathcal{S}$  to  $\text{H}_{\mathbb{H}}^{\leftarrow}(\mathcal{S}, I_0)$  factors naturally into*

$$-_a \text{Sph}_{\mathbb{H}} \rightarrow \text{DWeil}_a \rightarrow \text{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))$$

*Proof* The argument is similar or both claims, we prove only the first one. For a finite subfield  $k' \subset k$  we may pick a  $k'$ -structure on  $\mathcal{O}$ . Then  $I_0$  admits a  $k'$ -structure and, as such, is pure of weight zero. So, by the decomposition theorem ([1]), one has  $\text{H}_{\mathbb{G}}^{\leftarrow}(\mathcal{S}, I_0) \in \text{D P}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a(F))$ .

It remains to show that each perverse cohomology sheaf  $K$  of  $\zeta_a^{-1} \text{H}_{\mathbb{G}}^{\leftarrow}(\mathcal{S}, I_0)$  lies in the full subcategory  $\text{P}_{\mathbb{H}\mathbb{Q}\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))$  of  $\text{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Upsilon_a(F))$ .

By definition of the Hecke functors,  $H_{\mathbb{G}}^-(\mathcal{S}, I_0)$  is the extension by zero from  $\text{Char}(\Pi_a)$ , so  $\zeta_a(K)$  also satisfies this property. This yields a  $U_{\mathbb{H}_a}(\mathcal{O})$ -action on  $K$ .

To get a  $U_{\mathbb{H}_a}^-(\mathcal{O})$ -action on  $K$ , consider the commutative diagram of equivalences

$$\begin{array}{ccc} P_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Upsilon_a(F)) & \xrightarrow{\zeta_a} & P_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \\ \downarrow \zeta_{1,a} & \swarrow \text{Four}_{\psi} & \\ P_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(\Pi_a^* \otimes \Omega(F)), & & \end{array}$$

where  $\text{Four}_{\psi}$  is the complete Fourier transform, and  $\zeta_{1,a}$  is the corresponding partial one.

For  $v \in \Pi_a^* \otimes \Omega(F)$  write  $s_{\check{\Pi}}(v)$  for the composition

$$\wedge^2 U_a^*(F) \xrightarrow{\wedge^2 v} \wedge^2 M_a(F) \rightarrow A_a(F)$$

Write  $\text{Char}(\Pi_a^* \otimes \Omega) \subset \Pi_a^* \otimes \Omega(F)$  for the ind-subscheme of  $v$  such that  $s_{\check{\Pi}}(v) : C_a \otimes \wedge^2 U_a^* \rightarrow \Omega$  is regular. The  $U_{\mathbb{H}_a}^-(\mathcal{O})$ -equivariance of  $K$  is equivalent to the fact that  $\zeta_{1,a}(K)$  is the extension by zero from  $\text{Char}(\Pi_a^* \otimes \Omega)$ .

By Lemma 4, we have  $\text{Four}_{\psi} H_{\mathbb{G}}^-(\mathcal{S}, I_0) \xrightarrow{\sim} H_{\mathbb{G}}^-(\mathcal{S}, \check{I}_0)$ , where  $\check{I}_0 := \text{Four}_{\psi}(I_0)$  is the constant perverse sheaf on  $\Pi_0^* \otimes \Omega$  extended by zero to  $\Pi_0^* \otimes \Omega(F)$ . Clearly,  $H_{\mathbb{G}}^-(\mathcal{S}, \check{I}_0)$  is the extension by zero from  $\text{Char}(\Pi_a^* \otimes \Omega)$ , and our assertion follows.  $\square$

According to Proposition 3, in what follows we will write  $H_{\mathbb{G}}^-(\cdot, I_0) : {}_{-a}\text{Sph}_{\mathbb{G}} \rightarrow \text{DWeil}_a$  and  $H_{\mathbb{H}}^-(\cdot, I_0) : {}_{-a}\text{Sph}_{\mathbb{H}} \rightarrow \text{DWeil}_a$  for the corresponding functors. From Proposition 2 one derives the following.

**Corollary 2.** *For  $a \in \mathbb{Z}$ ,  $\mathcal{S} \in {}_{-a}\text{Sph}_{\mathbb{G}}$ ,  $\mathcal{T} \in {}_{-a}\text{Sph}_{\mathbb{H}}$  there are canonical isomorphisms*

$$\mathcal{F}_{\text{Weil}_a} H_{\mathbb{G}}^-(\mathcal{S}, I_0) \xrightarrow{\sim} H_{\mathbb{G}}^-(\mathcal{S}, S_{W_0(F)})$$

and

$$\mathcal{F}_{\text{Weil}_a} H_{\mathbb{H}}^-(\mathcal{T}, I_0) \xrightarrow{\sim} H_{\mathbb{H}}^-(\mathcal{T}, S_{W_0(F)})$$

in  $\text{DP}_{\mathcal{T}_a}(\tilde{\mathcal{L}}_d(W_a(F)))$ .

Thus, Theorem 3 is reduced to the following.

**Theorem 4.** *Let the maps  $\kappa$  be as in Theorem 3.*

1) *Assume  $m \leq n$ . The two functors  ${}_{-a}\text{Sph}_{\mathbb{G}} \rightarrow \text{DWeil}_a$  given by*

$$\mathcal{S} \mapsto H_{\mathbb{G}}^-(\mathcal{S}, I_0) \quad \text{and} \quad \mathcal{S} \mapsto H_{\mathbb{H}}^-(\text{gRes}^{\kappa}(\mathcal{S}), I_0)$$

*are isomorphic.*

2) *Assume  $m > n$ . The two functors  ${}_{-a}\text{Sph}_{\mathbb{H}} \rightarrow \text{DWeil}_a$  given by*

$$\mathcal{T} \mapsto H_{\mathbb{H}}^-(\mathcal{T}, I_0) \quad \text{and} \quad \mathcal{T} \mapsto H_{\mathbb{G}}^-(\text{gRes}^{\kappa}(*\mathcal{T}), I_0)$$

*are isomorphic.*

*Remark 6.* For  $a = 0$  Theorem 4 is nothing but ([7], Theorem 7).

#### 4.8.5 HECKE OPERATORS FOR LEVI SUBGROUPS

For  $a \in \mathbb{Z}$  set  $Q\Pi_a = U_a^* \otimes C_a \otimes L_a \subset \Pi_a$  and  $Q\Upsilon_a = L_a^* \otimes A_a \otimes U_a \subset \Upsilon_a$ .

We are going to define for  $a, a' \in \mathbb{Z}$  Hecke functors

$$H_{Q(\mathbb{G})}^{\leftarrow} : {}_{a'-a}\mathrm{Sph}_{Q(\mathbb{G})} \times D_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Pi_{a'}(F)) \rightarrow D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F)) \quad (18)$$

in a way compatible with the functors defined in Section 4.8.

Let  ${}^a\mathcal{X}Q\Pi$  be the stack classifying a  $Q(\mathbb{H})$ -torsor  $(U, C)$  over  $\mathrm{Spec} \mathcal{O}$ , a  $Q(\mathbb{G})$ -torsor  $(L, A)$  over  $\mathrm{Spec} \mathcal{O}$ , an isomorphism  $A \otimes C \xrightarrow{\sim} \Omega(a)$ , and a section  $s \in U^* \otimes C \otimes L(F)$ .

Informally, we think of  $D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F))$  as the derived category on  ${}^a\mathcal{X}Q\Pi$ . Consider the stack  ${}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Pi, Q(\mathbb{G})}$  classifying: a point of  ${}^a\mathcal{X}Q\Pi$  as above, a lattice  $L' \subset L(F)$ , for which we set  $A' = A(a' - a)$ . We get a diagram

$${}^a\mathcal{X}Q\Pi \xleftarrow{h^{\leftarrow}} {}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Pi, Q(\mathbb{G})} \xrightarrow{h^{\rightarrow}} {}^{a'}\mathcal{X}Q\Pi,$$

where  $h^{\leftarrow}$  sends the above collection to  $(U, C, L, A, s)$ , the map  $h^{\rightarrow}$  sends the above collection to  $(U, C, L', A', s')$ , where  $s'$  is the image of  $s$  under  $U^* \otimes C \otimes L(F) \xrightarrow{\sim} U^* \otimes C \otimes L'(F)$ .

Trivializing a point of  ${}^{a'}\mathcal{X}Q\Pi$  (resp., of  ${}^a\mathcal{X}Q\Pi$ ), one gets isomorphisms

$$\mathrm{id}^r : {}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Pi, Q(\mathbb{G})} \xrightarrow{\sim} (Q\Pi_{a'}(F) \times \mathrm{Gr}_{Q(\mathbb{G}_{a'})}^{a-a'})/Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})$$

and

$$\mathrm{id}^l : {}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Pi, Q(\mathbb{G})} \xrightarrow{\sim} (Q\Pi_a(F) \times \mathrm{Gr}_{Q(\mathbb{G}_a)}^{a'-a})/Q\mathbb{G}\mathbb{H}_a(\mathcal{O})$$

So, for

$$K \in D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F)), \quad K' \in D_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Pi_{a'}(F))$$

and  $\mathcal{S} \in {}_{a'-a}\mathrm{Sph}_{Q(\mathbb{G})}$ ,  $\mathcal{S}' \in {}_{a-a'}\mathrm{Sph}_{Q(\mathbb{G})}$  one can form their twisted exterior products  $(K \tilde{\boxtimes} \mathcal{S})^l$  and  $(K' \tilde{\boxtimes} \mathcal{S}')^r$  on  ${}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Pi, Q(\mathbb{G})}$ . The functor (18) is defined by

$$H_{Q(\mathbb{G})}^{\leftarrow}(\mathcal{S}', K') = h_!^{\leftarrow}(K' \tilde{\boxtimes} \mathcal{S}')^r$$

Let  ${}^a\mathcal{X}Q\Upsilon$  be the stack classifying a  $Q(\mathbb{H})$ -torsor  $(U, C)$  over  $\mathrm{Spec} \mathcal{O}$ , a  $Q(\mathbb{G})$ -torsor  $(L, A)$  over  $\mathrm{Spec} \mathcal{O}$ , an isomorphism  $A \otimes C \xrightarrow{\sim} \Omega(a)$ , and a section  $s \in U \otimes A \otimes L^*(F)$ . Informally, we think of  $D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$  as the derived category on  ${}^a\mathcal{X}Q\Upsilon$ . One defines the Hecke functor

$$H_{Q(\mathbb{H})}^{\leftarrow} : {}_{a'-a}\mathrm{Sph}_{Q(\mathbb{H})} \times D_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Upsilon_{a'}(F)) \rightarrow D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)) \quad (19)$$

using a similar diagram

$${}^a\mathcal{X}Q\Upsilon \xleftarrow{h^{\leftarrow}} {}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Upsilon, Q(\mathbb{H})} \xrightarrow{h^{\rightarrow}} {}^{a'}\mathcal{X}Q\Upsilon$$

By abuse of notation, we also write  $I_0$  for the constant perverse sheaf on  $Q\Upsilon_0$  and on  $Q\Pi_0$ , the exact meaning is easily understood from the context. The next result is a straightforward consequence of ([7], Corollary 4).

**Proposition 4.** 1) Assume  $m > n$ . The functor

$$-_a \text{Sph}_{Q(\mathbb{G})} \rightarrow \text{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F))$$

given by  $\mathcal{S} \mapsto \text{H}_{Q(\mathbb{G})}^-(\mathcal{S}, I_0)$  takes values in  $\text{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}^{ss}(Q\Pi_a(F))$  and induces an equivalence

$$-_a \text{Sph}_{Q(\mathbb{G})} \xrightarrow{\sim} \text{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}^{ss}(Q\Pi_a(F))$$

2) Assume  $m \leq n$ . The functor

$$-_a \text{Sph}_{Q(\mathbb{H})} \rightarrow \text{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$$

given by  $\mathcal{S} \mapsto \text{H}_{Q(\mathbb{H})}^-(\mathcal{S}, I_0)$  takes values in  $\text{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}^{ss}(Q\Upsilon_a(F))$  and induces an equivalence

$$-_a \text{Sph}_{Q(\mathbb{H})} \xrightarrow{\sim} \text{P}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}^{ss}(Q\Upsilon_a(F))$$

□

4.8.5.2 For  $a, a' \in \mathbb{Z}$  we will use in Section 4.8.9 the following Hecke functor

$$\text{H}_{Q(\mathbb{G})}^- : {}_{a'-a} \text{Sph}_{Q(\mathbb{G})} \times \text{D}_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Upsilon_{a'}(F)) \rightarrow \text{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)) \quad (20)$$

Consider the stack  ${}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Upsilon, Q(\mathbb{G})}$  classifying: a point  $(U, C, L, A, s)$  of  ${}^a\mathcal{X}Q\Upsilon$  as above, a lattice  $L' \subset L(F)$  for which we set  $A' = A(a' - a)$ . We get a diagram

$${}^a\mathcal{X}Q\Upsilon \xleftarrow{h^-} {}^{a,a'}\mathcal{H}_{\mathcal{X}Q\Upsilon, Q(\mathbb{G})} \xrightarrow{h^+} {}^{a'}\mathcal{X}Q\Upsilon,$$

where  $h^-$  sends the above collection to  $(U, C, L, A, s)$ , and  $h^+$  sends the same collection to  $(U, C, L', A', s')$ , where  $s'$  is the image of  $s$  under  $U \otimes A \otimes L^*(F) \xrightarrow{\sim} U \otimes A' \otimes L'^*(F)$ . The functor (20) is defined as in Section 4.8.5 for the above diagram.

The following is a consequence of ([7], Lemma 11).

**Lemma 6.** For  $\mathcal{S} \in {}_{a'-a} \text{Sph}_{Q(\mathbb{G})}$  the diagram of functors is canonically 2-commutative

$$\begin{array}{ccc} \text{D}_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Upsilon_{a'}(F)) & \xrightarrow{\text{Four}_\psi} & \text{D}_{Q\mathbb{G}\mathbb{H}_{a'}(\mathcal{O})}(Q\Pi_{a'}(F)) \\ \downarrow \text{H}_{Q(\mathbb{G})}^-(\mathcal{S}, \cdot) & & \downarrow \text{H}_{Q(\mathbb{G})}^-(\mathcal{S}, \cdot) \\ \text{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)) & \xrightarrow{\text{Four}_\psi} & \text{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F)) \end{array}$$

#### 4.8.6 WEAK JACQUET FUNCTORS

As in ([7], Section 4.7) for each  $a \in \mathbb{Z}$  we define the weak Jacquet functors

$$J_{P_{\mathbb{H}_a}}^*, J_{P_{\mathbb{H}_a}}^! : \text{D}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)) \rightarrow \text{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)) \quad (21)$$

and

$$J_{P_{\mathbb{G}_a}}^*, J_{P_{\mathbb{G}_a}}^! : \text{D}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \rightarrow \text{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F)) \quad (22)$$

Both definitions being similar, we recall the definition of (21) only.

For a free  $\mathcal{O}$ -module of finite type  $M$  and  $N, r \in \mathbb{Z}$  with  $N + r \geq 0$  write  ${}_{N,r}M = M(N)/M(-r)$ .

For  $N + r \geq 0$  consider the natural embedding  $i_{N,r} : {}_{N,r}Q\Upsilon_a \hookrightarrow {}_{N,r}\Upsilon_a$ . Set

$$PQ\mathbb{G}_a = \{g = (g_1, g_2) \in P_{\mathbb{H}_a} \times Q(\mathbb{G}_a) \mid g \in \mathcal{T}_a\},$$

this is a group scheme over  $\text{Spec } \mathcal{O}$ . We have a diagram of stack quotients

$$\begin{array}{ccc} PQ\mathbb{G}_a(\mathcal{O}/t^{N+r}) \backslash ({}_{N,r}Q\Upsilon_a) & \xrightarrow{i_{N,r}} & PQ\mathbb{G}_a(\mathcal{O}/t^{N+r}) \backslash ({}_{N,r}\Upsilon_a) \xrightarrow{p} \mathbb{H}Q\mathbb{G}_a(\mathcal{O}/t^{N+r}) \backslash ({}_{N,r}\Upsilon_a) \\ \downarrow q & & \\ Q\mathbb{G}\mathbb{H}_a(\mathcal{O}/t^{N+r}) \backslash ({}_{N,r}Q\Upsilon_a), & & \end{array}$$

where  $t \in \mathcal{O}$  is a uniformizer,  $p$  comes from the inclusion  $P_{\mathbb{H}_a} \subset \mathbb{H}_a$ , and  $q$  is the natural quotient map. First, define functors

$$J_{P_{\mathbb{H}_a}}^*, J_{P_{\mathbb{H}_a}}^! : D_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O}/t^{N+r})}({}_{N,r}\Upsilon_a) \rightarrow D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O}/t^{N+r})}({}_{N,r}Q\Upsilon_a) \quad (23)$$

by

$$q^* \circ J_{P_{\mathbb{H}_a}}^* [\dim. \text{rel}(q)] = i_{N,r}^* p^* [\dim. \text{rel}(p) - rnm]$$

$$q^* \circ J_{P_{\mathbb{H}_a}}^! [\dim. \text{rel}(q)] = i_{N,r}^! p^* [\dim. \text{rel}(p) + rnm]$$

Since

$$q^* [\dim. \text{rel}(q)] : D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O}/t^{N+r})}({}_{N,r}Q\Upsilon_a) \rightarrow D_{PQ\mathbb{G}_a(\mathcal{O}/t^{N+r})}({}_{N,r}Q\Upsilon_a)$$

is an equivalence (exact for the perverse t-structures), the functors (23) are well-defined. Further, (23) are compatible with the transition functors in the definition of the corresponding derived categories, so give rise to the functors (21) in the limit as  $N, r$  go to infinity. Note that for (21) we get  $\mathbb{D} \circ J_{P_{\mathbb{H}_a}}^* \xrightarrow{\sim} J_{P_{\mathbb{H}_a}}^! \circ \mathbb{D}$  naturally.

We identify  $\mathbb{H} \xrightarrow{\sim} \mathbb{H}_0$  and  $Q(\mathbb{H}) \xrightarrow{\sim} Q(\mathbb{H}_0)$ . Let  $\check{\mu}_{\mathbb{H}} = \det V_0$  and  $\check{\nu}_{\mathbb{H}} = \det U_0$  viewed as characters of  $\mathbb{H}$  or, equivalently, as cocharacters of the center  $Z(\check{Q}(\mathbb{H}))$  of the Langlands dual group  $\check{Q}(\mathbb{H})$  of  $Q(\mathbb{H})$ . Let  $\kappa_{\mathbb{H}} : \check{Q}(\mathbb{H}) \times \mathbb{G}_m \rightarrow \check{\mathbb{H}}$  be the map, whose first component is the natural inclusion of the Levi subgroup, and the second one is  $2(\check{\rho}_{\mathbb{H}} - \check{\rho}_{Q(\mathbb{H})}) + n(\check{\mu}_{\mathbb{H}} - \check{\nu}_{\mathbb{H}})$ . The corresponding geometric restriction functor is denoted  $\text{gRes}^{\kappa_{\mathbb{H}}}$ .

**Lemma 7.** *For  $a, a' \in \mathbb{Z}$  and  $\mathcal{S} \in {}_{a'-a}\text{Sph}_{\mathbb{H}}$ ,  $K \in D_{\mathbb{H}Q\mathbb{G}_{a'}(\mathcal{O})}(\Upsilon_{a'}(F))$  there is a filtration in the derived category  $D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$  on*

$$J_{P_{\mathbb{H}_a}}^* \mathbb{H}_{\mathbb{H}}^{\leftarrow}(\mathcal{S}, K)$$

*such that the corresponding graded object identifies with  $\mathbb{H}_{Q(\mathbb{H})}^{\leftarrow}(\text{gRes}^{\kappa_{\mathbb{H}}}(\mathcal{S}), J_{P_{\mathbb{H}_{a'}}}^*(K))$ .*

*Proof* The proof is quite similar to ([7], Lemma 10), we only have to determine the corresponding map  $\kappa$ . To do so, it suffices to perform the calculation for a particular  $K$ . Let  $I_{a'}$  be the constant perverse sheaf on  $\Upsilon_{a'}$  extended by zero to  $\Upsilon_{a'}(F)$ . Take  $K = I_{a'}$ .

For  $s_1, s_2 \geq 0$  let  ${}_{s_1, s_2} \text{Gr}_{\mathbb{H}_a} \subset \text{Gr}_{\mathbb{H}_a}$  be the closed subscheme of  $h\mathbb{H}_a(\mathcal{O}) \in \text{Gr}_{\mathbb{H}_a}$  such that

$$V_a(-s_1) \subset hV_a \subset V_a(s_2)$$

Assume that  $s_1, s_2$  are large enough so that  $\mathcal{S}$  is the extension by zero from  ${}_{s_1, s_2} \text{Gr}_{\mathbb{H}_a}$ . Then  $H_{\mathbb{H}}^{\leftarrow}(\mathcal{S}, I_{a'}) \in D_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(s_2, s_1 \Upsilon_a)$  is as follows. Write  ${}_{0, s_1} \Upsilon \tilde{\times}_{s_1, s_2} \text{Gr}_{\mathbb{H}_a}$  for the scheme classifying pairs

$$h\mathbb{H}_a(\mathcal{O}) \in {}_{s_1, s_2} \text{Gr}_{\mathbb{H}_a}, v \in L_a^* \otimes A_a \otimes (hV_a)/V_a(-s_1)$$

Let  $\pi : {}_{0, s_1} \Upsilon \tilde{\times}_{s_1, s_2} \text{Gr}_{\mathbb{H}_a} \rightarrow {}_{s_2, s_1} \Upsilon_a$  be the map sending  $(h\mathbb{H}_a(\mathcal{O}), v)$  to  $v$ . By definition,

$$H_{\mathbb{H}}^{\leftarrow}(\mathcal{S}, I_{a'}) \xrightarrow{\sim} \pi_!(\bar{\mathbb{Q}}_{\ell} \tilde{\boxtimes} \mathcal{S}), \quad (24)$$

where  $\bar{\mathbb{Q}}_{\ell} \tilde{\boxtimes} \mathcal{S}$  is normalized to be perverse. If  $\theta \in \pi_1(\mathbb{H})$  then  ${}_{0, s_1} \Upsilon \tilde{\times}_{s_1, s_2} \text{Gr}_{\mathbb{H}_a}$  is a vector bundle over  ${}_{s_1, s_2} \text{Gr}_{\mathbb{H}_a}^{\theta}$  of rank  $2s_1nm - \langle \theta, n\check{\mu}_{\mathbb{H}} \rangle$ .

Let  ${}_{s_1, s_2} P_{\mathbb{H}_a} = \{p \in P_{\mathbb{H}_a}(F) \mid V_a(-s_1) \subset pV_a \subset V_a(s_2)\}$ . Then

$${}_{s_1, s_2} \text{Gr}_{P_{\mathbb{H}_a}} = ({}_{s_1, s_2} P_{\mathbb{H}_a}(F))/P_{\mathbb{H}_a}(\mathcal{O})$$

is closed in  $\text{Gr}_{\mathbb{P}_{\mathbb{H}_a}}$ . The natural map  ${}_{s_1, s_2} \text{Gr}_{P_{\mathbb{H}_a}} \rightarrow {}_{s_1, s_2} \text{Gr}_{\mathbb{H}_a}$  at the level of reduced schemes yields a stratification of  ${}_{s_1, s_2} \text{Gr}_{\mathbb{H}_a}$  by the connected components of  ${}_{s_1, s_2} \text{Gr}_{P_{\mathbb{H}_a}}$ . Calculate (24) with respect to this stratification. Denote by  ${}_{s_1, s_2} \text{Gr}_{Q(\mathbb{H}_a)} \subset \text{Gr}_{Q(\mathbb{H}_a)}$  the closed subscheme of  $hQ(\mathbb{H}_a) \in \text{Gr}_{Q(\mathbb{H}_a)}$  satisfying

$$U_a(-s_1) \subset hU_a \subset U_a(s_2),$$

write  $\mathfrak{t}_P : \text{Gr}_{P_{\mathbb{H}_a}} \rightarrow \text{Gr}_{Q(\mathbb{H}_a)}$  for the natural map. We have the diagram

$$\begin{array}{ccccccc} {}_{0, s_1} Q\Upsilon \tilde{\times}_{s_1, s_2} \text{Gr}_{Q(\mathbb{H}_a)} & \xleftarrow{\text{id} \times \mathfrak{t}_P} & {}_{0, s_1} Q\Upsilon \tilde{\times}_{s_1, s_2} \text{Gr}_{P_{\mathbb{H}_a}} & \hookrightarrow & {}_{0, s_1} \Upsilon \tilde{\times}_{s_1, s_2} \text{Gr}_{P_{\mathbb{H}_a}} & \rightarrow & {}_{0, s_1} \Upsilon \tilde{\times}_{s_1, s_2} \text{Gr}_{\mathbb{H}_a} \\ & \searrow \pi_Q & \downarrow & & \downarrow & \swarrow \pi & \\ & & {}_{s_2, s_1} Q\Upsilon_a & \hookrightarrow & {}_{s_2, s_1} \Upsilon_a, & & \end{array}$$

where the square is cartesian. Here  ${}_{0, s_1} \Upsilon \tilde{\times}_{s_1, s_2} \text{Gr}_{P_{\mathbb{H}_a}}$  is the scheme classifying pairs

$$hP_{\mathbb{H}_a}(\mathcal{O}) \in {}_{s_1, s_2} \text{Gr}_{P_{\mathbb{H}_a}}, v \in L_a^* \otimes A_a \otimes (hV_a)/V_a(-s_1),$$

and  ${}_{0, s_1} Q\Upsilon \tilde{\times}_{s_1, s_2} \text{Gr}_{P_{\mathbb{H}_a}}$  is its closed subscheme given by the condition  $v \in L_a^* \otimes A_a \otimes (hU_a)/U_a(-s_1)$ .

By definition, for  $\mathcal{T} \in {}_{a'-a} \text{Sph}_{Q(\mathbb{H})}$  we have

$$H_{Q(\mathbb{H})}^{\leftarrow}(\mathcal{T}, I_{a'}) \xrightarrow{\sim} \pi_{Q!}(\bar{\mathbb{Q}}_{\ell} \tilde{\boxtimes} \mathcal{T}),$$

where  $\bar{\mathbb{Q}}_{\ell} \tilde{\boxtimes} \mathcal{T}$  is normalized to be perverse. If  $\theta \in \pi_1(Q(\mathbb{H}))$  then  ${}_{0, s_1} Q\Upsilon \tilde{\times}_{s_1, s_2} \text{Gr}_{Q(\mathbb{H}_a)}$  is a vector bundle over  ${}_{s_1, s_2} \text{Gr}_{Q(\mathbb{H}_a)}^{\theta}$  of rank  $s_1nm - \langle \theta, n\check{\nu}_{\mathbb{H}} \rangle$ . Our assertion follows.  $\square$

We identify  $\mathbb{G} \xrightarrow{\sim} \mathbb{G}_0$ ,  $Q(\mathbb{G}) \xrightarrow{\sim} Q(\mathbb{G}_0)$ . Write  $\check{\mu}_{\mathbb{G}} = \det M_0$  and  $\check{\nu}_{\mathbb{G}} = \det L_0$  as cocharacters of the center  $Z(\check{Q}(\mathbb{G}))$  of the Langlands dual group  $\check{Q}(\mathbb{G})$  of  $Q(\mathbb{G})$ . Let  $\kappa_{\mathbb{G}} : \check{Q}(\mathbb{G}) \times \mathbb{G}_m \rightarrow \check{\mathbb{G}}$  be the map whose first component is the natural inclusion of the Levi subgroup, and the second one is  $2(\check{\rho}_{\mathbb{G}} - \check{\rho}_{Q(\mathbb{G})}) + m(\check{\mu}_{\mathbb{G}} - \check{\nu}_{\mathbb{G}})$ . The corresponding geometric restriction functor is denoted  $\mathrm{gRes}^{\kappa_{\mathbb{G}}}$ .

**Lemma 8.** *For  $a, a' \in \mathbb{Z}$  and  $\mathcal{S} \in {}_{a'-a}\mathrm{Sph}_{\mathbb{G}}$ ,  $K \in \mathrm{D}_{\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})}(\Pi_{a'}(F))$  there is a filtration in the derived category  $\mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F))$  on*

$$J_{P_{\mathbb{G}_a}}^* \mathrm{H}_{\mathbb{G}}^{\leftarrow}(\mathcal{S}, K)$$

*such that the corresponding graded object identifies with  $\mathrm{H}_{Q(\mathbb{G})}^{\leftarrow}(\mathrm{gRes}^{\kappa_{\mathbb{G}}}(\mathcal{S}), J_{P_{\mathbb{G}_{a'}}}^*(K))$ .  $\square$*

We will use Lemmas 7 and 8 in the following form (the proof is as in [7], Corollary 3).

**Corollary 3.** *Let For  $a, a' \in \mathbb{Z}$  and  $\mathcal{S} \in {}_{a'-a}\mathrm{Sph}_{\mathbb{H}}$ . Assume that  $K \in \mathrm{P}_{\mathbb{H}Q\mathbb{G}_{a'}(\mathcal{O})}(\Upsilon_{a'}(F))$  admits a  $k_0$ -structure for some finite subfield  $k_0 \subset k$  and, as such, is pure of weight zero. Then  $J_{P_{\mathbb{H}_{a'}}}^*(K)$  is also pure of weight zero over  $k_0$ , and there is an isomorphism in  $\mathrm{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F))$*

$$J_{P_{\mathbb{H}_a}}^* \mathrm{H}_{\mathbb{H}}^{\leftarrow}(\mathcal{S}, K) \xrightarrow{\sim} \mathrm{H}_{Q(\mathbb{H})}^{\leftarrow}(\mathrm{gRes}^{\kappa_{\mathbb{H}}}(\mathcal{S}), J_{P_{\mathbb{H}_{a'}}}^*(K))$$

*(Similar strengthened version of Lemma 8 also holds.)*

#### 4.8.7 ACTION OF $\mathrm{Sph}_{\mathbb{G}}$

Pick a maximal torus and a Borel subgroup  $T_{\mathbb{G}} \subset B_{\mathbb{G}} \subset \mathbb{G}$ , and similarly for  $\mathbb{H}$ . We assume  $T_{\mathbb{G}} \subset Q(\mathbb{G})$  and  $T_{\mathbb{H}} \subset Q(\mathbb{H})$ . A trivialization of the  $\mathbb{G}_a$ -torsor  $(M_a, A_a)$  over  $\mathrm{Spec} \mathcal{O}$  yields a maximal torus and a Borel subgroup in  $\mathbb{G}_a$ , hence also an equivalence  $\mathrm{Sph}_{\mathbb{G}_a} \xrightarrow{\sim} \mathrm{Sph}_{\mathbb{G}}$  and a bijection  $\Lambda_{\mathbb{G}_a}^+ \xrightarrow{\sim} \Lambda_{\mathbb{G}}^+$  as in Section 4.3.2 (and similarly for  $\mathbb{H}$  and  $\mathbb{H}_a$ ).

Write  $\check{\omega}_i$  for the h.w. of the fundamental representation of  $\mathbb{G}_a$  that appear in  $\wedge^i M_a$  for  $i = 1, \dots, n$ . All the weights of  $\wedge^i M_a$  are  $\leq \check{\omega}_i$ . Write  $\check{\omega}_0$  for the h.w. of the  $\mathbb{G}_a$ -module  $A_a$ .

For  $\lambda \in \Lambda_{\mathbb{G}}^+$  set  $a = \langle \lambda, \check{\omega}_0 \rangle$  then  $\mathcal{A}_{\mathbb{G}}^{\lambda} \in {}_{-a}\mathrm{Sph}_{\mathbb{G}}$ . By definition, the complex

$$\mathrm{H}_{\mathbb{G}}^{\lambda}(I_0) = \mathrm{H}_{\mathbb{G}}^{\leftarrow}(\mathcal{A}^{\lambda}, I_0) \in \mathrm{D}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a(F))$$

is as follows. Set  $r = \langle \lambda, \check{\omega}_1 \rangle$  and  $N = \langle -w_0^{\mathbb{G}}(\lambda), \check{\omega}_1 \rangle$ . Let  ${}_{0,r}\Pi \tilde{\times} \overline{\mathrm{Gr}}_{\mathbb{G}_a}^{\lambda}$  be the scheme classifying  $g \in \overline{\mathrm{Gr}}_{\mathbb{G}_a}^{\lambda}$ ,  $x \in U_a^* \otimes C_a \otimes ((gM_a)/M_a(-r))$ . Let

$$\pi : {}_{0,r}\Pi \tilde{\times} \overline{\mathrm{Gr}}_{\mathbb{G}_a}^{\lambda} \rightarrow {}_{N,r}\Pi_a \tag{25}$$

be the map sending  $(x, g\mathbb{G}_a(\mathcal{O}))$  to  $x$ . Then  $\mathrm{H}_{\mathbb{G}}^{\lambda}(I_0) \xrightarrow{\sim} \pi_!(\bar{\mathbb{Q}}_{\ell} \boxtimes \tilde{\mathcal{A}}_{\mathbb{G}}^{\lambda})$  canonically (recall that  $\bar{\mathbb{Q}}_{\ell} \boxtimes \mathcal{A}_{\mathbb{G}}^{\lambda}$  is normalized to be perverse).

Define the closed subscheme  ${}_{\lambda}\Pi_a \subset \Pi_a(N)$  as follows. A point  $v \in \Pi_a(N)$  lies in  ${}_{\lambda}\Pi_a$  if the following conditions hold:

C1)  $v \in \text{Char}(\Pi_a)$ ;

C2) for  $i = 1, \dots, n$  the map  $\wedge^i v : \wedge^i(U_a \otimes C_a^{-1}) \rightarrow (\wedge^i M_a)(\langle -w_0^{\mathbb{G}}(\lambda), \check{\omega}_i \rangle)$  is regular.

The subscheme  $\lambda \Pi_a$  is stable under translations by  $\Pi_a(-r)$ , so there is a closed subscheme  $\lambda_{,N} \Pi_a \subset {}_{N,r} \Pi_a$  such that  $\lambda \Pi_a$  is the preimage of  $\lambda_{,N} \Pi_a$  under the projection  $\Pi_a(N) \rightarrow {}_{N,r} \Pi_a$ . Since all the weights of  $\wedge^i M_a$  are  $\leq \check{\omega}_i$ , the map (25) factors through the closed subscheme  $\lambda_{,N} \Pi_a \subset {}_{N,r} \Pi_a$ .

For each  $v \in \text{Char}(\Pi_a)$  let us define a  $\mathcal{O}$ -lattice  $M_v \subset M_a(F)$  as follows. View  $v$  as a map  $U_a \otimes C_a^{-1} \rightarrow M_a(F)$ . For a  $\mathcal{O}$ -lattice  $R \subset M_a(F)$  set

$$R^\perp = \{m \in M_a(F) \mid \langle m, x \rangle \in A_a(-a) \text{ for all } x \in R\}$$

Consider two cases.

CASE:  $a$  is even. For  $v \in \text{Char}(\Pi_a)$  set  $R_v = v(U_a \otimes C_a^{-1}) + M_a(-\frac{a}{2})$  and  $M_v = v(U_a \otimes C_a^{-1}) + R_v^\perp$ . Then  $R_v^\perp \subset M_v \subset R_v$ , and the induced form  $\wedge^2 M_v \rightarrow A_a(-a)$  is regular and nondegenerate. So,  $M_v \in \text{Gr}_{\mathbb{G}_a}^{-a}$ .

CASE:  $a$  is odd. Let  $b = (-a-1)/2$ . Note that  $(M_a(b))^\perp = M_a(b+1)$ . Set  $R_v = v(U_a \otimes C_a^{-1}) + M_a(b+1)$  and  $M_v = v(U_a \otimes C_a^{-1}) + R_v^\perp$ . Clearly, the induced form  $\wedge^2 M_v \rightarrow A_a(-a)$  is regular, but still can be degenerate. We call  $v$  *generic* if the form  $\wedge^2 M_v \rightarrow A_a(-a)$  is nondegenerate. In this case  $M_v \in \text{Gr}_{\mathbb{G}_a}^{-a}$ .

For  $a$  even we get a stratification of  $\text{Char}(\Pi_a)$  indexed by  $\{\lambda \in \Lambda_{\mathbb{G}}^+ \mid \langle \lambda, \check{\omega}_0 \rangle = a\}$ , the stratum  $\lambda \text{Char}(\Pi_a)$  is given by the condition that  $M_v \in \text{Gr}_{\mathbb{G}_a}^\lambda$ . This condition is also equivalent to requiring that there is an isomorphism of  $\mathcal{O}$ -modules

$$R_v/(M_a(-a/2)) \xrightarrow{\sim} \mathcal{O}/t^{a_1 - \frac{a}{2}} \oplus \dots \oplus \mathcal{O}/t^{a_n - \frac{a}{2}},$$

where  $t \in \mathcal{O}$  is a uniformizer.

Clearly,  $\lambda \text{Char}(\Pi_a) \subset \lambda \Pi_a$ . There is a unique open subscheme  $\lambda_{,N} \Pi_a^0 \subset \lambda_{,N} \Pi_a$  whose preimage under the projection  $\lambda \Pi_a \rightarrow \lambda_{,N} \Pi_a$  equals  $\lambda \text{Char}(\Pi_a)$ .

We say that a morphism of free  $\mathcal{O}$ -modules  $M_1 \rightarrow M_2$  is *maximal* if it does not factor through  $M_2(-1) \subset M_2$ .

For  $a$  odd define  $\lambda \text{Char}(\Pi_a) \subset \lambda \Pi_a$  as the open subscheme given by the condition that each map  $\wedge^i v$  in C2) is maximal. Then there is an open subscheme  $\lambda_{,N} \Pi_a^0 \subset \lambda_{,N} \Pi_a$  whose preimage under the projection  $\lambda \Pi_a \rightarrow \lambda_{,N} \Pi_a$  equals  $\lambda \text{Char}(\Pi_a)$ . One checks that any  $v \in \lambda \text{Char}(\Pi_a)$  is generic and the corresponding lattice  $M_v$  satisfies  $M_v \in \text{Gr}_{\mathbb{G}_a}^\lambda$ . Note that for  $v \in \lambda \text{Char}(\Pi_a)$  we have an isomorphism of  $\mathcal{O}$ -modules

$$R_v/(M_a(b+1)) \xrightarrow{\sim} \mathcal{O}/t^{a_1 - (a+1)/2} \oplus \dots \oplus \mathcal{O}/t^{a_n - (a+1)/2}$$

for any uniformizer  $t \in \mathcal{O}$ .

Write  $\text{IC}(\lambda_{,N} \Pi_a^0)$  for the intersection cohomology sheaf of  $\lambda_{,N} \Pi_a^0$ .



**Proposition 5.** Let  $\lambda \in \Lambda_{\mathbb{G}}^+$  with  $\langle \lambda, \check{\omega}_0 \rangle = a$ .

1) The map

$$\pi : {}_{0,r}\Pi \tilde{\times} \overline{\text{Gr}}_{\mathbb{G}_a}^\lambda \rightarrow {}_{\lambda,N}\Pi_a$$

is an isomorphism over the open subscheme  ${}_{\lambda,N}\Pi_a^0$ .

2) Assume  $m > n$  then one has a canonical isomorphism  $H_{\mathbb{G}}^\lambda(I_0) \xrightarrow{\sim} \text{IC}({}_{\lambda,N}\Pi_a^0)$ .

*Proof* 1) The fibre of  $\pi$  over  $v \in {}_{\lambda,N}\Pi_a^0$  is the scheme classifying lattices  $M' \in \overline{\text{Gr}}_{\mathbb{G}_a}^\lambda$  such that  $v(U_a \otimes C_a^{-1}) \subset M'$ . Given such a lattice  $M'$  let us show that  $M_v = M'$ .

Consider first the case of  $a$  odd. The inclusion  $R_v \subset M' + M_a(b+1)$  must be an equality, because for  $M' \in \text{Gr}_{\mathbb{G}_a}^\mu$  with  $\mu \leq \lambda$  we have

$$\dim(M' + M_a(b+1))/(M_a(b+1)) = \epsilon(\mu) \leq \epsilon(\lambda) = \dim R_v/(M_a(b+1))$$

We have denoted here  $\epsilon(\mu) = \langle \mu, \check{\omega}_n \rangle - \frac{n}{2}(a+1)$ . So,  $M_v = v(U_a \otimes C_a^{-1}) + (M' \cap M_a(b)) \subset M'$  is also an equality, because both  $M_v$  and  $M'$  have symplectic forms with values in  $A_a(-a)$ .

The case of  $a$  even is quite similar to ([7], Lemma 15). Namely, the inclusion  $R_v \subset M' + M_a(-\frac{a}{2})$  must be an equality, because for  $M' \in \text{Gr}_{\mathbb{G}_a}^\mu$  with  $\mu \leq \lambda$  we get

$$\dim(M' + M_a(-a/2))/(M_a(-a/2)) = \epsilon(\mu) \leq \epsilon(\lambda) = \dim R_v/(M_a(-a/2))$$

Here for  $a$  even we have set  $\epsilon(\mu) = \langle \mu, \check{\omega}_n \rangle - \frac{n}{2}a$ . So,

$$M_v = v(U_a \otimes C_a^{-1}) + (M' \cap (M_a(-a/2))) \subset M'$$

is also an equality. The first assertion follows.

2) For  $m \geq n$  the scheme  ${}_{\lambda,N}\Pi_a^0$  is nonempty, so  $\text{IC}({}_{\lambda,N}\Pi_a^0)$  appears in  $H_{\mathbb{G}}^\lambda(I_0)$  with multiplicity one. So, it suffices to show that

$$\text{Hom}(H_{\mathbb{G}}^\lambda(I_0), H_{\mathbb{G}}^\lambda(I_0)) = \bar{\mathbb{Q}}_\ell,$$

where  $\text{Hom}$  is taken in the derived category  $D_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F))$ . By adjointness,

$$\text{Hom}(H_{\mathbb{G}}^\lambda(I_0), H_{\mathbb{G}}^\lambda(I_0)) \xrightarrow{\sim} \text{Hom}(H_{\mathbb{G}}^{-w_0^{\mathbb{G}}(\lambda)} H_{\mathbb{G}}^\lambda(I_0), I_0),$$

where  $\text{Hom}$  in the RHS is taken in  $D_{\mathbb{G}Q\mathbb{H}_0(\mathcal{O})}(\Pi_0(F))$ . We are reduced to show that for any  $0 \neq \mu \in \Lambda_{\mathbb{G}}^+$  with  $\langle \mu, \check{\omega}_0 \rangle = 0$  one has

$$\text{Hom}(H_{\mathbb{G}}^\mu(I_0), I_0) = 0$$

in  $D_{\mathbb{G}Q\mathbb{H}_0(\mathcal{O})}(\Pi_0(F))$ . The latter assertion is true for  $m > n$ , it is proved in ([7], part 2) of Lemma 15).  $\square$

*Remark 7.* For any  $a, b \in \mathbb{Z}$  let us construct an equivalence  $\text{Weil}_a \xrightarrow{\sim} \text{Weil}_{a+2b}$ . Pick isomorphisms of  $\mathcal{O}$ -modules

$$L_a(b) \xrightarrow{\sim} L_{a+2b}, \quad A_a(2b) \xrightarrow{\sim} A_{a+2b}, \quad U_a \xrightarrow{\sim} U_{a+2b}, \quad (26)$$

They yield isomorphisms  $C_a \xrightarrow{\sim} C_{a+2b}$ ,  $V_a \xrightarrow{\sim} V_{a+2b}$ ,  $M_a(b) \xrightarrow{\sim} M_{a+2b}$ . Hence, also isomorphisms  $Q(\mathbb{G}_a) \xrightarrow{\sim} Q(\mathbb{G}_{a+2b})$ ,  $\mathbb{G}_a \xrightarrow{\sim} \mathbb{G}_{a+2b}$  of group schemes over  $\text{Spec } \mathcal{O}$  (and similarly for  $\mathbb{H}$ ). We also get isomorphisms of group schemes over  $\text{Spec } \mathcal{O}$

$$Q\mathbb{G}\mathbb{H}_a \xrightarrow{\sim} Q\mathbb{G}\mathbb{H}_{a+2b}, \quad \mathbb{G}Q\mathbb{H}_a \xrightarrow{\sim} \mathbb{G}Q\mathbb{H}_{a+2b}, \quad \mathbb{H}Q\mathbb{G}_a \xrightarrow{\sim} \mathbb{H}Q\mathbb{G}_{a+2b}$$

The isomorphisms (26) also yield  $\Pi_a(b) \xrightarrow{\sim} \Pi_{a+2b}$  and  $\Upsilon_a(b) \xrightarrow{\sim} \Upsilon_{a+2b}$ . In turn, we get equivalences

$$\text{P}_{\mathbb{H}Q\mathbb{G}_a}(\Upsilon_a(F)) \xrightarrow{\sim} \text{P}_{\mathbb{H}Q\mathbb{G}_{a+2b}}(\Upsilon_{a+2b}(F)), \quad \text{P}_{\mathbb{G}Q\mathbb{H}_a}(\Pi_a(F)) \xrightarrow{\sim} \text{P}_{\mathbb{G}Q\mathbb{H}_{a+2b}}(\Pi_{a+2b}(F))$$

which yield the desired equivalence  $\text{Weil}_a \xrightarrow{\sim} \text{Weil}_{a+2b}$ . The diagram commutes

$$\begin{array}{ccc} -a \text{ Sph}_{\mathbb{G}} & \rightarrow & \text{Weil}_a \\ \downarrow \epsilon & & \downarrow \wr \\ -a-2b \text{ Sph}_{\mathbb{G}} & \rightarrow & \text{Weil}_{a+2b}, \end{array}$$

where the horizontal arrows are given by  $\mathcal{S} \mapsto \text{H}_{\mathbb{G}}^{\leftarrow}(\mathcal{S}, I_0)$ , and  $\epsilon$ , at the level of representations of  $\check{G}$ , is given by  $V \mapsto V \otimes V^{b\omega}$ . Here  $V^{\omega}$  is the one-dimensional representation of  $\check{G}$  with h.w.  $\omega$  such that  $\langle \omega, \check{\omega}_0 \rangle = 2$ . So, the case of  $a$  even in Proposition 5 also follows from ([7], Lemma 15).

4.8.7.2 Let  $k_0 \subset k$  be a finite subfield. In this subsection we assume that all the objects of Sections 4 are defined over  $k_0$ .

Write  $\text{Weil}_{a,k_0}$  for the category of triples  $(\mathcal{F}_1, \mathcal{F}_2, \beta)$  as in Definition 2 of  $\text{Weil}_a$  but with a  $k_0$ -structure and, as such, pure of weight zero. It is understood that the Fourier transform functors are normalized to preserve purity. Note that for any  $(\mathcal{F}_1, \mathcal{F}_2, \beta) \in \text{Weil}_a$  the perverse sheaf  $\mathcal{F}_1$  is  $\mathbb{G}_m$ -equivariant with respect to the homotheties on  $\Upsilon_a(F)$ .

Denote by  $\text{DWeil}_{a,k_0}$  the category of complexes as in the definition of  $\text{DWeil}_a$  but, in addition, with a  $k_0$ -structure and, as such, pure of weight zero. So, for an object of  $\text{DWeil}_{a,k_0}$  its semi-simplification is a bounded complex of the form  $\bigoplus_{i \in \mathbb{Z}} F_i[i] \left(\frac{i}{2}\right)$  with  $F_i \in \text{Weil}_{a,k_0}$ .

Write  $F_0$  for  $k_0$ -valued points of  $F$ . For a totally disconnected locally compact space  $Y$  write  $\mathcal{S}(Y)$  for the Schwarz space of locally constant  $\bar{\mathbb{Q}}_{\ell}$ -valued functions on  $Y$  with compact support. Write  $\text{Weil}_a(k_0)$  for the  $\bar{\mathbb{Q}}_{\ell}$ -vector space of pairs  $(\mathcal{F}_1, \mathcal{F}_2)$ , where  $\mathcal{F}_1 \in \mathcal{S}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F_0))$ ,  $\mathcal{F}_2 \in \mathcal{S}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F_0))$  with  $\zeta_a(\mathcal{F}_1) = \mathcal{F}_2$ .

Write  $\mathcal{P}$  for the composition of functors

$$\text{DWeil}_a \xrightarrow{f_{\mathbb{H}}} \text{DP}_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)) \xrightarrow{J_{P_{\mathbb{H}_a}}^*} \text{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)),$$

where  $f_{\mathbb{H}}$  sends  $(\mathcal{F}_1, \mathcal{F}_2, \beta)$  to  $\mathcal{F}_1$ . By abuse of notation, we also write  $\mathcal{P} : \text{DWeil}_{a,k_0} \rightarrow \text{D}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F_0))$  for the similarly defined functor over  $k_0$ .

**Proposition 6.** *For  $i = 1, 2$  let  $K_i \in \mathrm{DWeil}_{a,k_0}$ . If  $\mathcal{P}(K_1) \xrightarrow{\sim} \mathcal{P}(K_2)$  then  $K_1 \xrightarrow{\sim} K_2$  in  $\mathrm{DWeil}_a$ .*

*Proof* Write  $K_{k_0}$  (resp.,  $DK_{k_0}$ ) for the Grothendieck group of the category  $\mathrm{Weil}_{a,k_0}$  (resp., of  $\mathrm{DWeil}_{a,k_0}$ ). Note that  $DK_{k_0} \xrightarrow{\sim} K_{k_0} \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$ . Write  $\Upsilon K_{k_0}$  for the Grothendieck group of the category of pure complexes of weight zero on  $Q\Upsilon_a(F_0)$ , whose all perverse cohomologies lie in  $P_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F_0))$ . The functor  $J_{P_{\mathbb{H}_a}}^*$  yields a homomorphism  $J_{P_{\mathbb{H}_a}}^* : DK_{k_0} \rightarrow \Upsilon K_{k_0}$ . Let us show that it is injective. Let  $F$  be an objects in its kernel. For any finite subfield  $k_0 \subset k_1 \subset k$  the map  $\mathrm{tr}_{k_1}$  trace of Frobenius over  $k_1$  fits into the diagram

$$\begin{array}{ccc} DK_{k_0} & \xrightarrow{J_{P_{\mathbb{H}_a}}^*} & \Upsilon K_{k_0} \\ \downarrow \mathrm{tr}_{k_1} & & \downarrow \mathrm{tr}_{k_1} \\ \mathrm{Weil}_a(k_1) & \xrightarrow{J_{k_1}} & \mathcal{S}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F_1)), \end{array} \quad (27)$$

where  $F_1$  denotes the  $k_1$ -valued points of  $F$ . By (Lemma 11, Appendix A),  $J_{k_1}$  is injective, so  $\mathrm{tr}_{k_1}(F) = 0$  for any finite extension  $k_0 \subset k_1$ . By the result of Laumon ([4], Theorem 1.1.2) this implies  $F = 0$  in  $DK_{k_0}$ . Finally, if  $K_1 = K_2$  in  $DK_{k_0}$  then  $K_1 \xrightarrow{\sim} K_2$  in  $\mathrm{DWeil}_a$ .  $\square$

The following result will not be used in this paper, its proof is found in Appendix A.

**Proposition A.1.** *Assume  $m > n$ . The map  $K_0(-_a \mathrm{Sph}_{\mathbb{G}}) \otimes \bar{\mathbb{Q}}_{\ell} \rightarrow \mathrm{Weil}_a(k_0)$  given by  $\mathcal{S} \mapsto \mathrm{tr}_{k_0} H_{\mathbb{G}}^-(\mathcal{S}, I_0)$  is an isomorphism of  $\bar{\mathbb{Q}}_{\ell}$ -vector spaces.*

Write  $\mathrm{Weil}_a^{ss} \subset \mathrm{Weil}_a$  for the full subcategory of semi-simple objects.

**Conjecture 1.** *Assume  $m > n$ . The functor  ${}_a \mathrm{Sph}_{\mathbb{G}} \rightarrow \mathrm{Weil}_a^{ss}$  given by  $\mathcal{S} \mapsto H_{\mathbb{G}}^-(\mathcal{S}, I_0)$  is an equivalence of categories.*

#### 4.8.8 ACTION OF $\mathrm{Sph}_{\mathbb{H}}$

We write  $V^{\check{\lambda}}$  for the irreducible  $\mathbb{H}$ -module with h.w.  $\check{\lambda}$ . Assume that  $V_0$  is a  $2m$ -dimensional  $k$ -vector space with nondegenerate symmetric form  $\mathrm{Sym}^2 V_0 \rightarrow C_0$ , and  $\mathbb{H}$  is the connected component of unity of  $\mathrm{GO}(V_0)$ . Write  $\check{\alpha}_0$  for the h.w. of the  $\mathbb{H}$ -module  $C_0$ . For  $0 < i < m$  let  $\check{\alpha}_i$  denote the h.w. of the irreducible  $\mathbb{H}$ -module  $\wedge^i V_0$ . Remind that

$$\wedge^m V_0 \xrightarrow{\sim} V^{\check{\alpha}_m} \oplus V^{\check{\alpha}'_m}$$

is a direct sum of two irreducible representations, this is our definition of  $\check{\alpha}_m, \check{\alpha}'_m$ . Say that a maximal isotropic subspace  $\mathcal{L} \subset V_0$  is  $\check{\alpha}_m$ -oriented (resp.,  $\check{\alpha}'_m$ -oriented) if  $\wedge^m \mathcal{L} \subset V^{\check{\alpha}_m}$  (resp.,  $\wedge^m \mathcal{L} \subset V^{\check{\alpha}'_m}$ ). The group  $\mathbb{H}$  has two orbits on the set of maximal isotropic subspaces in  $V_0$  given by the orientation.

Remind that  $\mathrm{Gr}_{\mathbb{H}}^b$  classifies lattices  $V' \subset V_0(F)$  such that the induced form  $\mathrm{Sym}^2 V' \rightarrow C(b)$  is regular and nondegenerate, here  $C = C_0(\mathcal{O})$ .

Let  $\lambda \in \Lambda_{\mathbb{H}}^+$ , set  $a = \langle \lambda, \check{\alpha}_0 \rangle$ . Remind that  $\mathcal{A}_{\mathbb{H}}^{\lambda} \in \mathrm{Sph}_{\mathbb{H}}$  denotes the IC-sheaf of  $\overline{\mathrm{Gr}}_{\mathbb{H}}^{\lambda}$ , so  $\mathcal{A}_{\mathbb{H}}^{\lambda} \in {}_a \mathrm{Sph}_{\mathbb{H}}$ . By definition, the complex

$$H_{\mathbb{H}}^{\lambda}(I_0) = H_{\mathbb{H}}^-(\mathcal{A}_{\mathbb{H}}^{\lambda}, I_0) \in \mathrm{D}_{\mathbb{H}Q\mathbb{G}_a}(\Upsilon_a(F))$$

is as follows. Set  $r = \langle \lambda, \check{\alpha}_1 \rangle$  and  $N = \langle -w_0^{\mathbb{H}}(\lambda), \check{\alpha}_1 \rangle$ . Let  ${}_{0,r}\Upsilon \tilde{\times} \overline{\text{Gr}}_{\mathbb{H}_a}^\lambda$  be the scheme classifying  $h \in \overline{\text{Gr}}_{\mathbb{H}_a}^\lambda$ ,  $x \in L_a^* \otimes A_a \otimes ((hV_a)/V_a(-r))$ . Let

$$\pi : {}_{0,r}\Upsilon \tilde{\times} \overline{\text{Gr}}_{\mathbb{H}_a}^\lambda \rightarrow {}_{N,r}\Upsilon_a \quad (28)$$

be the map sending  $(x, h\mathbb{H}_a(\mathcal{O}))$  to  $x$ . Then  $H_{\mathbb{H}}^\lambda(I_0) \xrightarrow{\sim} \pi_!(\bar{\mathbb{Q}}_\ell \boxtimes \tilde{\mathcal{A}}_{\mathbb{H}}^\lambda)[b]$  canonically, where  $b$  is the unique integer such that  $\bar{\mathbb{Q}}_\ell \boxtimes \tilde{\mathcal{A}}_{\mathbb{H}}^\lambda[b]$  is perverse.

View a point of  $\Upsilon_a(F)$  as a map  $L_a \otimes A_a^* \rightarrow V_a(F)$ . Define a closed subscheme  ${}_\lambda\Upsilon_a \subset \Upsilon_a(N)$  as follows. A point  $v \in \Upsilon_a(N)$  lies in  ${}_\lambda\Upsilon_a$  if the following conditions hold:

- C1)  $v \in \text{Char}(\Upsilon_a)$ ;
- C2) for  $1 \leq i < m$  the map  $\wedge^i v : \wedge^i(L_a \otimes A_a^*) \rightarrow (\wedge^i V_a)(\langle -w_0^{\mathbb{H}}(\lambda), \check{\alpha}_i \rangle)$  is regular;
- C3) the map  $(v_m, v'_m) : \wedge^m(L_a \otimes A_a^*) \rightarrow V_a^{\check{\alpha}_m}(\langle -w_0^{\mathbb{H}}(\lambda), \check{\alpha}_m \rangle) \oplus V_a^{\check{\alpha}'_m}(\langle -w_0^{\mathbb{H}}(\lambda), \check{\alpha}'_m \rangle)$  induced by  $\wedge^m v$  is regular.

The scheme  ${}_\lambda\Upsilon_a$  is stable under translations by  $\Upsilon_a(-r)$ , so there is a closed subscheme  ${}_{\lambda,N}\Upsilon_a \subset {}_{N,r}\Upsilon_a$  such that  ${}_\lambda\Upsilon_a$  is the preimage of  ${}_{\lambda,N}\Upsilon_a$  under the projection  $\Upsilon_a(N) \rightarrow {}_{N,r}\Upsilon_a$ . Clearly, the map (28) factors through the closed subscheme  ${}_{\lambda,N}\Upsilon_a \subset {}_{N,r}\Upsilon_a$ .

For each  $v \in \text{Char}(\Upsilon_a)$  define a  $\mathcal{O}$ -lattice  $V_v \subset V_a(F)$  as follows. For a  $\mathcal{O}$ -lattice  $R \subset V_a(F)$  set

$$R^\perp = \{x \in V_a(F) \mid \langle x, y \rangle \in C_a(-a) \text{ for all } y \in R\}$$

Consider two cases.

CASE:  $a$  is even. For  $v \in \text{Char}(\Upsilon_a)$  set  $R_v = v(L_a \otimes A_a^*) + V_a(-\frac{a}{2})$  and  $V_v = v(L_a \otimes A_a^*) + R_v^\perp$ . Then  $V_v \in \text{Gr}_{\mathbb{H}}^{-a}$ . In this case we get a stratification of  $\text{Char}(\Upsilon_a)$  by locally closed subschemes  ${}_\lambda\text{Char}(\Upsilon_a)$  indexed by  $\{\lambda \in \Lambda_{\mathbb{H}}^+ \mid \langle \lambda, \check{\alpha}_0 \rangle = a\}$ . Namely,  $v \in \text{Char}(\Upsilon_a)$  lies in  ${}_\lambda\text{Char}(\Upsilon_a)$  iff  $V_v \in \text{Gr}_{\mathbb{H}}^\lambda$ .

Clearly,  ${}_\lambda\text{Char}(\Upsilon_a) \subset {}_\lambda\Upsilon_a$ . There is a unique open subscheme  ${}_{\lambda,N}\Upsilon_a^0 \subset {}_{\lambda,N}\Upsilon_a$  whose preimage under the projection  ${}_\lambda\Upsilon_a \rightarrow {}_{\lambda,N}\Upsilon_a$  equals  ${}_{\lambda,N}\Upsilon_a^0$ .

CASE:  $a$  is odd. Let  $b = (-a - 1)/2$ . We have  $(V_a(b + 1))^\perp = V_a(b)$ . Set  $R_v = v(L_a \otimes A_a^*) + V_a(b + 1)$  and  $V_v = v(L_a \otimes A_a^*) + R_v^\perp$ . Then the induced form  $\text{Sym}^2 V_v \rightarrow C_a(-a)$  is regular, but still can be degenerate. We call  $v$  *generic* if the form  $\text{Sym}^2 V_v \rightarrow C_a(-a)$  is nondegenerate. In this case  $V_v \in \text{Gr}_{\mathbb{H}}^{-a}$ .

For  $a$  odd define an open subscheme  ${}_\lambda\text{Char}(\Upsilon_a) \subset {}_\lambda\Upsilon_a$  as follows. Note that  $\langle w_0^{\mathbb{H}}(\lambda), \check{\alpha}_m - \check{\alpha}'_m \rangle \neq 0$ . A point  $v \in {}_\lambda\Upsilon_a$  lies in  ${}_\lambda\text{Char}(\Upsilon_a)$  if the following conditions hold:

- the maps in C2) are maximal;
- if  $\langle w_0^{\mathbb{H}}(\lambda), \check{\alpha}_m - \check{\alpha}'_m \rangle < 0$  then  $v_m$  in C3) is maximal, otherwise  $v'_m$  in C3) is maximal.

There is a unique open subscheme  ${}_{\lambda,N}\Upsilon_a^0 \subset {}_{\lambda,N}\Upsilon_a$  whose preimage under the projection  ${}_{\lambda}\Upsilon_a \rightarrow {}_{\lambda,N}\Upsilon_a$  equals  ${}_{\lambda}\text{Char}(\Upsilon_a)$ .

Write  $\text{IC}({}_{\lambda,N}\Upsilon_a^0)$  for the intersection cohomology sheaf of  ${}_{\lambda,N}\Upsilon_a^0$ .

**Proposition 7.** *Let  $\lambda \in \Lambda_{\mathbb{H}}^+$  with  $\langle \lambda, \check{\alpha}_0 \rangle = a$ .*

1) *The map*

$$\pi : {}_{0,r}\Upsilon \times \overline{\text{Gr}}_{\mathbb{H}_a}^{\lambda} \rightarrow {}_{\lambda,N}\Upsilon_a$$

*is an isomorphism over the open subscheme  ${}_{\lambda,N}\Upsilon_a^0$ .*

2) *Assume  $m \leq n$  then one has a canonical isomorphism  $H_{\mathbb{H}}^{\lambda}(I_0) \xrightarrow{\sim} \text{IC}({}_{\lambda,N}\Upsilon_a^0)$ .*

*Proof* 1) Let  $v \in {}_{\lambda,N}\Upsilon_a^0$ . The fibre of  $\pi$  over  $v$  is the scheme classifying lattices  $V' \in \overline{\text{Gr}}_{\mathbb{H}_a}^{\lambda}$  such that  $v(L_a \otimes A_a^*) \subset V'$ . Given such a lattice  $V'$  let us show that  $V_v = V'$ .

In view of Remark 7 the case of  $a$  even is reduced to the case  $a = 0$ , and the latter is done in ([7], Lemma 14).

Consider the case of  $a$  odd. The inclusion  $R_v \subset V' + V_a(b+1)$  must be an equality, because for  $V' \in \text{Gr}_{\mathbb{H}_a}^{\mu}$  with  $\mu \leq \lambda$  we have

$$\dim(V' + V_a(b+1))/(V_a(b+1)) = \epsilon(\mu) \leq \epsilon(\lambda) = \dim R_v/(V_a(b+1))$$

We have denoted here  $\epsilon(\mu) = -m(b+1) + \max\{\langle -w_0^{\mathbb{H}}(\mu), \check{\alpha}_m \rangle, \langle -w_0^{\mathbb{H}}(\mu), \check{\alpha}'_m \rangle\}$ .

It follows that  $V_v = v(L_a \otimes A_a^{-1}) + (V' \cap V_a(b)) \subset V'$ . To prove that  $V' = V_v$ , it suffices to show that  $v$  is generic. This follows from the fact that  $(v(L_a \otimes A_a^{-1}) + R_v^{\perp})/R_v^{\perp}$  is a maximal isotropic subspace in  $R_v/R_v^{\perp}$ .

2) For  $m \leq n$  the scheme  ${}_{\lambda,N}\Upsilon_a^0$  is nonempty, so  $\text{IC}({}_{\lambda,N}\Upsilon_a^0)$  appears in  $H_{\mathbb{H}}^{\lambda}(I_0)$  with multiplicity one. Now it remains to show that

$$\text{Hom}(H_{\mathbb{H}}^{\lambda}(I_0), H_{\mathbb{H}}^{\lambda}(I_0)) = \bar{\mathbb{Q}}_{\ell},$$

where  $\text{Hom}$  is taken in the derived category  $D_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F))$ . By adjointness,

$$\text{Hom}(H_{\mathbb{H}}^{\lambda}(I_0), H_{\mathbb{H}}^{\lambda}(I_0)) \xrightarrow{\sim} \text{Hom}(H_{\mathbb{H}}^{-w_0^{\mathbb{H}}(\lambda)} H_{\mathbb{H}}^{\lambda}(I_0), I_0),$$

where  $\text{Hom}$  in the RHS is taken in  $D_{\mathbb{H}Q\mathbb{G}_0(\mathcal{O})}(\Upsilon_0(F))$ . We are reduced to show that for any  $0 \neq \mu \in \Lambda_{\mathbb{H}}^+$  with  $\langle \mu, \check{\alpha}_0 \rangle = 0$  one has

$$\text{Hom}(H_{\mathbb{H}}^{\mu}(I_0), I_0) = 0$$

in  $D_{\mathbb{H}Q\mathbb{G}_0(\mathcal{O})}(\Upsilon_0(F))$ . For  $m \leq n$  this is proved in ([7], part 2) of Lemma 14).  $\square$

As in the case  $m > n$ , assume for a moment that  $k_0 \subset k$  is a finite subfield, and all the objects introduced in Section 4 have a  $k_0$ -structure. The following result is analogous to Proposition A.1, its proof is omitted.

**Proposition A.2.** *Assume  $m \leq n$ . Then the map  $K_0(-_a \text{Sph}_{\mathbb{H}}) \otimes \bar{\mathbb{Q}}_{\ell} \rightarrow \text{Weil}_a(k_0)$  given by  $\mathcal{S} \mapsto \text{tr}_{k_0} H_{\mathbb{H}}^{\leftarrow}(\mathcal{S}, I_0)$  is an isomorphism of  $\bar{\mathbb{Q}}_{\ell}$ -vector spaces.*

**Conjecture 2.** Assume  $m \leq n$ . The functor  ${}_a \text{Sph}_{\mathbb{H}} \rightarrow \text{Weil}_a^{ss}$  given by  $\mathcal{S} \mapsto H_{\mathbb{H}}^{\leftarrow}(\mathcal{S}, I_0)$  is an equivalence of categories.

#### 4.8.9 Proof of Theorem 4

Use the notations of Section 4.8.7 and 4.8.8. Assume that  $U_0$  is  $\check{\alpha}_m$ -oriented. Below we identify  $\check{\omega}_0 : \mathbb{G}_m \xrightarrow{\sim} \check{\text{GL}}(A_0)$  and  $\check{\alpha}_0 : \mathbb{G}_m \xrightarrow{\sim} \check{\text{GL}}(C_0)$ .

For  $m > n$  fix a decomposition  $U_0 \xrightarrow{\sim} {}_1U \oplus {}_2U$  into direct sum of free  $\mathcal{O}$ -modules, where  ${}_1U$  is of rank  $n$ , and  ${}_2U$  is of rank  $m - n$ , fix also an isomorphism  ${}_1U \xrightarrow{\sim} L_0$  of  $\mathcal{O}$ -modules. We assume that these choices are compatible with the maximal tori chosen before. For  $m > n$  let  $\kappa_0 : \check{\text{GL}}(L_0) \times \mathbb{G}_m \rightarrow \check{\text{GL}}(U_0)$  be the composition

$$\check{\text{GL}}(L_0) \times \mathbb{G}_m \xrightarrow{\tau \times \text{id}} \check{\text{GL}}(L_0) \times \mathbb{G}_m = \check{\text{GL}}({}_1U) \times \mathbb{G}_m \xrightarrow{\text{id} \times 2\check{\rho}_{\text{GL}({}_2U)}} \check{\text{GL}}({}_1U) \times \check{\text{GL}}({}_2U) \xrightarrow{\text{Levi}} \check{\text{GL}}(U_0),$$

where  $\tau$  is an automorphism of  $\check{\text{GL}}(L_0)$  inducing the functor  $* : \text{Sph}_{\text{GL}(L_0)} \xrightarrow{\sim} \text{Sph}_{\text{GL}(L_0)}$ .

Let  $\kappa_Q : \check{Q}(\mathbb{G}) \times \mathbb{G}_m \rightarrow \check{Q}(\mathbb{H})$  be the map

$$\check{\text{GL}}(L_0) \times \check{\text{GL}}(A_0) \times \mathbb{G}_m \rightarrow \check{\text{GL}}(U_0) \times \check{\text{GL}}(C_0)$$

given by  $(x, y, z) \mapsto (\kappa_0(x, z), y\omega_m(x))$ . Here  $\omega_m$  is the unique coweight of the center of  $\text{GL}(L_0)$  such that  $\langle \omega_m, \check{\omega}_1 \rangle = 1$ .

Write  $\kappa_{Q,ex} : \check{Q}(\mathbb{G}) \times \mathbb{G}_m \rightarrow \check{Q}(\mathbb{H}) \times \mathbb{G}_m$  for the map  $(\kappa_Q, \text{pr})$ , where  $\text{pr} : \check{Q}(\mathbb{G}) \times \mathbb{G}_m \rightarrow \mathbb{G}_m$  is the projection.

For  $m \leq n$  fix a decomposition  $L_0 \xrightarrow{\sim} {}_1L \oplus {}_2L$  into direct sum of free  $\mathcal{O}$ -modules, where  ${}_1L$  is of rank  $m$ , and  ${}_2L$  is of rank  $n - m$ , fix also an isomorphism  $U_0 \xrightarrow{\sim} {}_1L$  of  $\mathcal{O}$ -modules. We assume that these choices are compatible with the maximal tori chosen before. For  $m \leq n$  we let  $\kappa_0 : \check{\text{GL}}(U_0) \times \mathbb{G}_m \rightarrow \check{\text{GL}}(L_0)$  be the composition

$$\check{\text{GL}}(U_0) \times \mathbb{G}_m = \check{\text{GL}}({}_1L) \times \mathbb{G}_m \xrightarrow{\text{id} \times 2\check{\rho}_{\text{GL}({}_2L)}} \check{\text{GL}}({}_1L) \times \check{\text{GL}}({}_2L) \xrightarrow{\text{Levi}} \check{\text{GL}}(L_0) \xrightarrow{\tau} \check{\text{GL}}(L_0),$$

here  $\tau$  is an automorphism inducing the functor  $* : \text{Sph}_{\text{GL}(L_0)} \xrightarrow{\sim} \text{Sph}_{\text{GL}(L_0)}$ .

Let  $\kappa_Q : \check{Q}(\mathbb{H}) \times \mathbb{G}_m \rightarrow \check{Q}(\mathbb{G})$  be the map

$$\check{\text{GL}}(U_0) \times \check{\text{GL}}(C_0) \times \mathbb{G}_m \rightarrow \check{\text{GL}}(L_0) \times \check{\text{GL}}(A_0)$$

given by  $(x, y, z) \mapsto (\kappa_0(x, z), y\alpha_m(x))$ . Here  $\alpha_m$  is the unique coweight of the center of  $\text{GL}(U_0)$  such that  $\langle \alpha_m, \check{\alpha}_1 \rangle = 1$ .

Define  $\kappa_{Q,ex} : \check{Q}(\mathbb{H}) \times \mathbb{G}_m \rightarrow \check{Q}(\mathbb{G}) \times \mathbb{G}_m$  as  $(\kappa_Q, \text{pr})$ . The following is a consequence of ([7], Corollary 5).

**Proposition 8.** 1) For  $m > n$  the two functors  ${}_a \text{Sph}_{Q(\mathbb{H})} \rightarrow D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$  given by

$$\mathcal{T} \mapsto H_{Q(\mathbb{H})}^{\leftarrow}(\mathcal{T}, I_0) \quad \text{and} \quad \mathcal{T} \mapsto H_{Q(\mathbb{G})}^{\leftarrow}(\text{gRes}^{\kappa_Q}(\mathcal{T}), I_0)$$

are isomorphic.

2) For  $m \leq n$  the two functors  $-_a \text{Sph}_{Q(\mathbb{G})} \rightarrow D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$  given by

$$\mathcal{T} \mapsto H_{Q(\mathbb{G})}^-(\mathcal{T}, I_0) \quad \text{and} \quad \mathcal{T} \mapsto H_{Q(\mathbb{H})}^-(\text{gRes}^{\kappa_Q}(\mathcal{T}), I_0)$$

are isomorphic.  $\square$

As in ([7], Theorem 7), for each  $a \in \mathbb{Z}$  the diagram of functors is canonically 2-commutative

$$\begin{array}{ccc} & D\text{Weil}_a & \\ \swarrow f_{\mathbb{H}} & & \searrow f_{\mathbb{G}} \\ DP_{\mathbb{H}Q\mathbb{G}_a(\mathcal{O})}(\Upsilon_a(F)) & & DP_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F)) \\ \downarrow J_{P_{\mathbb{H}_a}}^* & & \downarrow J_{P_{\mathbb{G}_a}}^* \\ D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F)) & \xrightarrow{\text{Four}_\psi} & D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F)), \end{array}$$

where  $f_{\mathbb{H}}$  (resp.,  $f_{\mathbb{G}}$ ) sends  $(\mathcal{F}_1, \mathcal{F}_2, \beta)$  to  $\mathcal{F}_1$  (resp., to  $\mathcal{F}_2$ ).

Remind the maps  $\kappa_{\mathbb{H}} : \check{Q}(\mathbb{H}) \times \mathbb{G}_m \rightarrow \check{H}$  and  $\kappa_{\mathbb{G}} : \check{Q}(\mathbb{G}) \times \mathbb{G}_m \rightarrow \check{\mathbb{G}}$  from Section 4.8.6. The restriction of  $\kappa_{\mathbb{H}}$  and of  $\kappa_{\mathbb{G}}$  to  $\mathbb{G}_m$  equals

$$2(\check{\rho}_{\mathbb{H}} - \check{\rho}_{Q(\mathbb{H})}) + nm\check{\alpha}_0 - n\check{\alpha}_m$$

and  $2(\check{\rho}_{\mathbb{G}} - \check{\rho}_{Q(\mathbb{G})}) + mn\check{\omega}_0 - m\check{\omega}_n$  respectively. From definitions one gets

$$\begin{cases} 2(\check{\rho}_{\mathbb{H}} - \check{\rho}_{Q(\mathbb{H})}) = (m-1)\check{\alpha}_m - \frac{m(m-1)}{2}\check{\alpha}_0 \\ 2(\check{\rho}_{\mathbb{G}} - \check{\rho}_{Q(\mathbb{G})}) = (n+1)\check{\omega}_n - \frac{n(n+1)}{2}\check{\omega}_0 \end{cases}$$

Write  $\kappa_{\mathbb{H},ex} : \check{Q}(\mathbb{H}) \times \mathbb{G}_m \rightarrow \check{H} \times \mathbb{G}_m$  for the map, whose first component is  $\kappa_{\mathbb{H}}$  and the second  $\check{Q}(\mathbb{H}) \times \mathbb{G}_m \rightarrow \mathbb{G}_m$  is the projection, and similarly for  $\kappa_{\mathbb{G},ex}$ .

By Corollary 3, for  $\mathcal{T} \in -_a \text{Sph}_{\mathbb{H}}$  and  $\mathcal{S} \in -_a \text{Sph}_{\mathbb{G}}$  we get isomorphisms

$$\mathcal{P}(H_{\mathbb{H}}^-(\mathcal{T}, I_0)) \xrightarrow{\sim} H_{Q(\mathbb{H})}^-(\text{gRes}^{\kappa_{\mathbb{H}}}(\mathcal{T}), I_0) \quad (29)$$

and

$$\mathcal{P}(H_{\mathbb{G}}^-(\mathcal{S}, I_0)) \xrightarrow{\sim} H_{Q(\mathbb{G})}^-(\text{gRes}^{\kappa_{\mathbb{G}}}(\mathcal{S}), I_0) \quad (30)$$

in  $D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$ . The Hecke functors in the RHS of (29) and (30) are from  $D_{Q\mathbb{G}\mathbb{H}_0(\mathcal{O})}(Q\Upsilon_0(F))$  to the category  $D_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F))$ .

CASE  $m > n$ . Proposition 8 together with (29) yields an isomorphism

$$\mathcal{P}(H_{\mathbb{H}}^-(\mathcal{T}, I_0)) \xrightarrow{\sim} H_{Q(\mathbb{G})}^-(\text{gRes}^{\kappa_Q, ex} \text{gRes}^{\kappa_{\mathbb{H}}}(\mathcal{T}), I_0)$$

We will define an automorphism  $\tau_{\mathbb{H}}$  of  $\check{\mathbb{H}}$  inducing  $* : \text{Rep}(\check{\mathbb{H}}) \xrightarrow{\sim} \text{Rep}(\check{\mathbb{H}})$  and  $\kappa$  making the following diagram commutative

$$\begin{array}{ccc} \check{\mathbb{G}} \times \mathbb{G}_m & \xrightarrow{\tau_{\mathbb{H}} \circ \kappa} & \check{\mathbb{H}} \\ \uparrow \kappa_{\mathbb{G}, ex} & & \uparrow \kappa_{\mathbb{H}} \\ \check{Q}(\mathbb{G}) \times \mathbb{G}_m & \xrightarrow{\kappa_Q, ex} & \check{Q}(\mathbb{H}) \times \mathbb{G}_m \end{array} \quad (31)$$

The above diagram together with (30) yield isomorphisms

$$\mathcal{P}(\mathrm{H}_{\mathbb{G}}^{\leftarrow}(\mathrm{gRes}^{\kappa}(*\mathcal{T}), I_0)) \xrightarrow{\sim} \mathrm{H}_{Q(\mathbb{G})}^{\leftarrow}(\mathrm{gRes}^{\kappa_{\mathbb{G}}} \mathrm{gRes}^{\kappa}(*\mathcal{T}), I_0) \xrightarrow{\sim} \mathrm{H}_{Q(\mathbb{G})}^{\leftarrow}(\mathrm{gRes}^{\kappa_{Q,ex}} \mathrm{gRes}^{\kappa_{\mathbb{H}}}(\mathcal{T}), I_0)$$

Thus, we get an isomorphism

$$\mathcal{P}(\mathrm{H}_{\mathbb{G}}^{\leftarrow}(\mathrm{gRes}^{\kappa}(*\mathcal{T}), I_0)) \xrightarrow{\sim} \mathcal{P}(\mathrm{H}_{\mathbb{H}}^{\leftarrow}(\mathcal{T}, I_0))$$

By Proposition 6, it lifts to the desired isomorphism in  $\mathrm{DWeil}_a$

$$\mathrm{H}_{\mathbb{G}}^{\leftarrow}(\mathrm{gRes}^{\kappa}(*\mathcal{T}), I_0) \xrightarrow{\sim} \mathrm{H}_{\mathbb{H}}^{\leftarrow}(\mathcal{T}, I_0)$$

Note that  $m \geq 2$ . Let  $W_0 = \bar{\mathbb{Q}}_{\ell}^m$ , let  $W_1$  (resp.,  $W_2$ ) be the subspace of  $W_1$  spanned by the first  $n$  (resp., last  $m - n$ ) base vectors. Equip  $W_0 \oplus W_0^*$  with the symmetric form given by the matrix

$$\begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix},$$

where  $E_m \in \mathrm{GL}(\bar{\mathbb{Q}}_{\ell})$  is the unity. Let  $i_{\mathbb{H}} \in \mathrm{Spin}(W_0 \oplus W_0^*)$  be the unique central element such that

$$\mathrm{SO}(W_0 \oplus W_0^*) \xrightarrow{\sim} \mathrm{Spin}(W_0 \oplus W_0^*) / \{i_{\mathbb{H}}\}$$

Realize  $\check{\mathbb{H}}$  as  $\mathrm{GSpin}(W_0 \oplus W_0^*) := \mathbb{G}_m \times \mathrm{Spin}(W_0 \oplus W_0^*) / \{(-1, i_{\mathbb{H}})\}$ . There is a unique automorphism  $\tau'$  of  $\mathrm{Spin}(W_0 \oplus W_0^*)$  that preserves  $i_{\mathbb{H}}$  and induces the automorphism  $g \mapsto {}^t g^{-1}$  on  $\mathrm{SO}(W_0 \oplus W_0^*)$ . The automorphism  $(a, g) \mapsto (a^{-1}, \tau'(g))$  of  $\mathbb{G}_m \times \mathrm{Spin}(W_0 \oplus W_0^*)$  descends to an automorphism of  $\check{\mathbb{H}}$  that we denote  $\tau_{\mathbb{H}}$ .

Let  $\bar{W} \subset W_2 \oplus W_2^*$  be the subspace spanned by  $e_{n+1} + e_{n+1}^*$ . Equip  $W_1 \oplus W_1^* \oplus \bar{W}$  with the induced form. Write  $i_{\mathbb{G}}$  for the central element of  $\mathrm{Spin}(W_1 \oplus W_1^* \oplus \bar{W})$ . Realize  $\check{\mathbb{G}}$  as

$$\mathrm{GSpin}(W_1 \oplus W_1^* \oplus \bar{W}) := \mathbb{G}_m \times \mathrm{Spin}(W_1 \oplus W_1^* \oplus \bar{W}) / \{(-1, i_{\mathbb{G}})\}$$

There is a unique inclusion  $\epsilon_0 : \mathrm{Spin}(W_1 \oplus W_1^* \oplus \bar{W}) \hookrightarrow \mathrm{Spin}(W_0 \oplus W_0^*)$  extending the natural inclusion  $\mathrm{SO}(W_1 \oplus W_1^* \oplus \bar{W}) \hookrightarrow \mathrm{SO}(W_0 \oplus W_0^*)$  and sending  $i_{\mathbb{G}}$  to  $i_{\mathbb{H}}$ . The map

$$\mathrm{id} \times \epsilon_0 : \mathbb{G}_m \times \mathrm{Spin}(W_1 \oplus W_1^* \oplus \bar{W}) \rightarrow \mathbb{G}_m \times \mathrm{Spin}(W_0 \oplus W_0^*)$$

gives rise to an inclusion  $i_{\kappa} : \check{\mathbb{G}} \hookrightarrow \check{\mathbb{H}}$ . Finally, there is a unique  $\alpha_{\kappa} : \mathbb{G}_m \rightarrow \check{\mathbb{H}}$  such that for  $\kappa := (i_{\kappa}, \alpha_{\kappa})$  the diagram (31) commutes. The map  $\tau_{\mathbb{H}} \circ \kappa : \check{T}_{\mathbb{G}} \rightarrow \check{T}_{\mathbb{H}}$  is uniquely defined by the formulas

$$\begin{cases} \check{\omega}_i \mapsto -\check{\alpha}_i + i\check{\alpha}_0, & 1 \leq i \leq n \\ \check{\omega}_0 \mapsto \check{\alpha}_0 \end{cases}$$

Using these formulas, one checks that

$$\tau_{\mathbb{H}}(\alpha_{\kappa}) = 2\check{\rho}_{\mathrm{GL}(2U)} + (m - 1 - n)(\check{\alpha}_m - \check{\alpha}_n) + \left(\frac{n(n+1)}{2} - \frac{m(m-1)}{2} - (n+1-m)n\right)\check{\alpha}_0$$

If  $m = n + 1$  then  $\alpha_{\kappa}$  is trivial.



CASE  $m \leq n$ . Proposition 8 together with (30) yields an isomorphism

$$\mathcal{P}(\mathcal{H}_{\mathbb{G}}^-(\mathcal{S}, I_0)) \xrightarrow{\sim} \mathcal{H}_{Q(\mathbb{H})}^-(\text{gRes}^{\kappa_Q, ex} \text{gRes}^{\kappa_{\mathbb{G}}}(\mathcal{S}), I_0)$$

We will define an automorphism  $\tau_{\mathbb{H}}$  of  $\check{\mathbb{H}}$  inducing  $* : \text{Rep}(\check{\mathbb{H}}) \xrightarrow{\sim} \text{Rep}(\check{\mathbb{H}})$  and  $\kappa$  making the following diagram commutative

$$\begin{array}{ccc} \check{\mathbb{H}} \times \mathbb{G}_m & \xrightarrow{\tau_{\mathbb{H}} \times \text{id}} & \check{\mathbb{H}} \times \mathbb{G}_m \xrightarrow{\kappa} \check{\mathbb{G}} \\ \uparrow \kappa_{\mathbb{H}, ex} & & \uparrow \kappa_{\mathbb{G}} \\ \check{Q}(\mathbb{H}) \times \mathbb{G}_m & \xrightarrow{\kappa_{Q, ex}} & \check{Q}(\mathbb{G}) \times \mathbb{G}_m \end{array} \quad (32)$$

The above diagram together with (29) yield isomorphisms

$$\mathcal{P}(\mathcal{H}_{\mathbb{H}}^-(\text{gRes}^{\kappa}(\mathcal{S}), I_0)) \xrightarrow{\sim} \mathcal{H}_{Q(\mathbb{H})}^-(\text{gRes}^{\kappa_{\mathbb{H}}}(\text{gRes}^{\kappa}(\mathcal{S})), I_0) \xrightarrow{\sim} \mathcal{H}_{Q(\mathbb{H})}^-(\text{gRes}^{\kappa_Q, ex} \text{gRes}^{\kappa_{\mathbb{G}}}(\mathcal{S}), I_0)$$

Thus, we get

$$\mathcal{P}(\mathcal{H}_{\mathbb{H}}^-(\text{gRes}^{\kappa}(\mathcal{S}), I_0)) \xrightarrow{\sim} \mathcal{P}(\mathcal{H}_{\mathbb{G}}^-(\mathcal{S}, I_0))$$

By Proposition 6, it lifts to the desired isomorphism in  $\text{DWeil}_a$

$$\mathcal{H}_{\mathbb{H}}^-(\text{gRes}^{\kappa}(\mathcal{S}), I_0) \xrightarrow{\sim} \mathcal{H}_{\mathbb{G}}^-(\mathcal{S}, I_0)$$

For  $m = 1$  the map  $\kappa_{\mathbb{H}, ex}$  is an isomorphism, so there is a unique  $\kappa$  making (32) commutative. Now assume  $m > 1$ . Let  $W_0 = \bar{\mathbb{Q}}_{\ell}^n$ , let  $W_1$  (resp.,  $W_2$ ) be the subspace of  $W_0$  generated by the first  $m$  (resp., last  $n - m$ ) vectors. Equip  $W_0 \oplus W_0 \oplus \bar{\mathbb{Q}}_{\ell}$  with the symmetric form given by the matrix

$$\begin{pmatrix} 0 & E_n & 0 \\ E_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where  $E_n \in \text{GL}_n(\bar{\mathbb{Q}}_{\ell})$  is the unity. Write  $i_{\mathbb{G}}$  for nontrivial the central element of  $\text{Spin}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_{\ell})$ . Realize  $\check{\mathbb{G}}$  as

$$\text{GSpin}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_{\ell}) := \mathbb{G}_m \times \text{Spin}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_{\ell}) / \{(-1, i_{\mathbb{G}})\}$$

Equip the subspace  $W_1 \oplus W_1^* \subset W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_{\ell}$  with the induced symmetric form. Write  $i_{\mathbb{H}}$  for the unique central element of  $\text{Spin}(W_1 \oplus W_1^*)$  such that  $\text{SO}(W_1 \oplus W_1^*) \xrightarrow{\sim} \text{Spin}(W_1 \oplus W_1^*) / \{i_{\mathbb{H}}\}$ . Realize  $\check{\mathbb{H}}$  as

$$\text{GSpin}(W_1 \oplus W_1^*) := \mathbb{G}_m \times \text{Spin}(W_1 \oplus W_1^*) / \{(-1, i_{\mathbb{H}})\}$$

There is a unique automorphism  $\tau'$  of  $\text{Spin}(W_1 \oplus W_1^*)$  preserving  $i_{\mathbb{H}}$  and inducing the map  $g \mapsto {}^t g^{-1}$  on  $\text{SO}(W_1 \oplus W_1^*)$ . The automorphism  $(a, g) \mapsto (a^{-1}, \tau'(g))$  of  $\mathbb{G}_m \times \text{Spin}(W_1 \oplus W_1^*)$  descends to an automorphism of  $\check{\mathbb{H}}$  that we denote  $\tau_{\mathbb{H}}$ .

There is a unique inclusion

$$\epsilon_0 : \text{Spin}(W_1 \oplus W_1^*) \hookrightarrow \text{Spin}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_{\ell})$$

sending  $i_{\mathbb{H}}$  to  $i_{\mathbb{G}}$  and extending the natural inclusion  $\mathrm{SO}(W_1 \oplus W_1^*) \hookrightarrow \mathrm{SO}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell)$ . The map

$$\mathrm{id} \times \epsilon_0 : \mathbb{G}_m \times \mathrm{Spin}(W_1 \oplus W_1^*) \rightarrow \mathbb{G}_m \times \mathrm{Spin}(W_0 \oplus W_0^* \oplus \bar{\mathbb{Q}}_\ell)$$

gives rise to an inclusion  $i_\kappa : \check{\mathbb{H}} \rightarrow \check{\mathbb{G}}$ . Now there is a unique  $\alpha_\kappa : \mathbb{G}_m \rightarrow \check{\mathbb{G}}$  such that for  $\kappa = (i_\kappa, \alpha_\kappa)$  the diagram (32) commutes. The map  $\kappa \circ \tau_{\mathbb{H}} : \check{T}_{\mathbb{H}} \rightarrow \check{T}_{\mathbb{G}}$  is uniquely defined by the formulas

$$\begin{cases} \check{\alpha}_i \mapsto -\check{\omega}_i + i\check{\omega}_0, & 1 \leq i \leq m \\ \check{\alpha}_0 \mapsto \check{\omega}_0 \end{cases}$$

From this formulas one gets that

$$\alpha_\kappa = (n+1-m)(\check{\omega}_n - \check{\omega}_m) + (mn - \frac{m(m-1)}{2} - \frac{n(n+1)}{2})\check{\omega}_0 - 2\check{\rho}_{GL(2L)}$$

In particular,  $\alpha_\kappa$  is trivial for  $m = n$ . Theorem 4 is proved.

## 5. GLOBAL THEORY

5.1 In Sections 5.1-5.2 we derive Theorem 2 from Theorem 3. To simplify notations, fix a closed point  $\tilde{x} \in \tilde{X}$ . Let  ${}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}$  be obtained from  ${}^a\mathrm{Bun}_{G,\tilde{H}}$  by the base change  $\tilde{x} \rightarrow \tilde{X}$ . We will establish isomorphisms (4) and (5) over  ${}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}$ . The fact that these isomorphisms depend on  $\tilde{x}$  as expected is left to the reader. Set  $x = \pi(\tilde{x})$ .

Recall the line bundle  $\mathcal{E}$  from Section 2.3, we have  $\pi^*\mathcal{E} \xrightarrow{\sim} \mathcal{O}_{\tilde{X}}$  canonically. So, the above choice of  $\tilde{x}$  yields a trivialization  $\mathcal{E} \xrightarrow{\sim} \mathcal{O}|_{D_x}$  over  $D_x = \mathrm{Spec} \mathcal{O}_x$ . The corresponding trivialization for  $\sigma\tilde{x}$  is the previous one multiplied by  $-1$ . We will apply Theorem 3 for  $\mathcal{O} = \mathcal{O}_x$ .

Recall the stack  ${}^a\mathcal{X}\mathcal{L}$  and a line bundle  ${}^a\mathcal{A}_{\mathcal{X}\mathcal{L}}$  on it introduced in Section 4.2. A point of  ${}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}$  is given by a collection:  $(M, \mathcal{A}) \in \mathrm{Bun}_G$ ,  $(V, \mathcal{C}) \in \mathrm{Bun}_{\tilde{H}}$ , and an isomorphism  $\mathcal{A} \otimes \mathcal{C} \xrightarrow{\sim} \Omega(ax)$ . Let  ${}^a\xi : {}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}} \rightarrow {}^a\mathcal{X}\mathcal{L}$  be the map sending  $(M, \mathcal{A}, V, \mathcal{C})$  to  $(M, \mathcal{A}, V, \mathcal{C})|_{D_x}$  together with the discrete lagrangian subspace  $L = H^0(X - x, M \otimes V) \subset M \otimes V(F_x)$ .

**Lemma 9.** *For a point  $(\mathcal{M}, \mathcal{A}, \mathcal{V}, \mathcal{C})$  of  ${}^{a\tilde{x}}\mathrm{Bun}_{G,\mathbb{H}}$  there is a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism*

$$\det \mathrm{R}\Gamma(X, \mathcal{M} \otimes \mathcal{V}) \otimes \mathcal{C}_x^{-anm} \xrightarrow{\sim} \frac{\det \mathrm{R}\Gamma(X, \mathcal{M})^{2m} \otimes \det \mathrm{R}\Gamma(X, \mathcal{V})^{2n}}{\det \mathrm{R}\Gamma(X, \mathcal{C})^{2nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{2nm}}$$

Here  $\mathcal{C}_x$  is of parity zero as  $\mathbb{Z}/2\mathbb{Z}$ -graded.

*Proof* By ([5], Lemma 1), we get a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\det \mathrm{R}\Gamma(X, \mathcal{M} \otimes \mathcal{V}) \xrightarrow{\sim} \frac{\det \mathrm{R}\Gamma(X, \mathcal{M})^{2m} \otimes \det \mathrm{R}\Gamma(X, \mathcal{V})^{2n}}{\det \mathrm{R}\Gamma(X, \mathcal{A}^n) \otimes \det \mathrm{R}\Gamma(X, \det \mathcal{V})} \otimes \frac{\det \mathrm{R}\Gamma(X, \mathcal{A}^n \otimes \det \mathcal{V})}{\det \mathrm{R}\Gamma(X, \mathcal{O})^{4nm-1}}$$

Applying this to  $\mathcal{M} = \mathcal{O}^n \oplus \mathcal{A}^n$  with natural symplectic form  $\wedge^2 \mathcal{M} \rightarrow \mathcal{A}$ , we get

$$\frac{\det \mathrm{R}\Gamma(X, \mathcal{V} \otimes \mathcal{A})^n}{\det \mathrm{R}\Gamma(X, \mathcal{V})^n} \xrightarrow{\sim} \frac{\det \mathrm{R}\Gamma(X, \mathcal{A})^{2nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{A}^n \otimes \det \mathcal{V})}{\det \mathrm{R}\Gamma(X, \mathcal{A}^n) \otimes \det \mathrm{R}\Gamma(X, \det \mathcal{V}) \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{2nm-1}}$$

Since  $\mathcal{A} \otimes \mathcal{C} \xrightarrow{\sim} \Omega(ax)$  and  $\mathcal{V} \xrightarrow{\sim} \mathcal{V}^* \otimes \mathcal{C}$ , the LHS of the above formula identifies with

$$\det \mathrm{R}\Gamma(X, V/V(-ax))^{-n} \xrightarrow{\sim} (\det V_x)^{-an} \otimes \det(\mathcal{O}/\mathcal{O}(-ax))^{-2mn}$$

We have used a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism

$$\det(V/V(-ax)) \xrightarrow{\sim} (\det V_x)^a \otimes (\det(\mathcal{O}/\mathcal{O}(-ax)))^{2m}$$

Since  $\det V_x \xrightarrow{\sim} \mathcal{C}_x^m$ , we get

$$\det \mathrm{R}\Gamma(X, \mathcal{M} \otimes \mathcal{V}) \xrightarrow{\sim} \frac{\det \mathrm{R}\Gamma(X, \mathcal{M})^{2m} \otimes \det \mathrm{R}\Gamma(X, \mathcal{V})^{2n}}{\det \mathrm{R}\Gamma(X, \mathcal{A})^{2nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{2nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{O}/\mathcal{O}(-ax))^{2nm}} \otimes \mathcal{C}_x^{-anm}$$

To simplify the above expression, note that  $\det \mathrm{R}\Gamma(X, \mathcal{A}) \xrightarrow{\sim} \det \mathrm{R}\Gamma(X, \mathcal{C}(-ax))$  and

$$\det \mathrm{R}\Gamma(X, \mathcal{C}) \xrightarrow{\sim} \det \mathrm{R}\Gamma(X, \mathcal{C}(-ax)) \otimes \mathcal{C}_x^a \otimes \det \mathrm{R}\Gamma(X, \mathcal{O}/\mathcal{O}(-ax))$$

Our assertion follows.  $\square$

Let  ${}^a\mathcal{A}$  be the line bundle on  ${}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}$  with fibre  $\det \mathrm{R}\Gamma(X, M \otimes V) \otimes \mathcal{C}_x^{-anm}$  at  $(M, \mathcal{A}, V, \mathcal{C})$ . We have canonically  $({}^a\xi)^*({}^a\mathcal{A}_{\mathcal{X}\mathcal{L}}) \xrightarrow{\sim} {}^a\mathcal{A}$ . Extend  ${}^a\xi$  to a morphism  ${}^a\tilde{\xi} : {}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}} \rightarrow {}^a\widetilde{\mathcal{X}\mathcal{L}}$  sending  $(M, \mathcal{A}, V, \mathcal{C})$  to its image under  ${}^a\xi$  together with the one-dimensional space

$$\mathcal{B} = \frac{\det \mathrm{R}\Gamma(X, \mathcal{M})^m \otimes \det \mathrm{R}\Gamma(X, \mathcal{V})^n}{\det \mathrm{R}\Gamma(X, \mathcal{C})^{nm} \otimes \det \mathrm{R}\Gamma(X, \mathcal{O})^{nm}}$$

equipped with the isomorphism  $\mathcal{B}^2 \xrightarrow{\sim} \det \mathrm{R}\Gamma(X, \mathcal{M} \otimes \mathcal{V}) \otimes \mathcal{C}_x^{-anm}$  of Lemma 9.

5.2 Let  ${}^{a\tilde{x}}\mathcal{H}_{G,\tilde{H}}$  be the stack classifying collections: a point of the Hecke stack  $(M, \mathcal{A}, M', \mathcal{A}', \beta) \in {}_x\mathcal{H}_G$  such that the isomorphism  $\beta$  of the  $G$ -torsors  $(M, \mathcal{A})$  and  $(M', \mathcal{A}')$  over  $X - x$  induces an isomorphism  $\mathcal{A}(-ax) \xrightarrow{\sim} \mathcal{A}'$ ; a  $\tilde{H}$ -torsor  $(V, \mathcal{C}) \in \mathrm{Bun}_{\tilde{H}}$ , and an isomorphism  $\mathcal{A}' \otimes \mathcal{C} \xrightarrow{\sim} \Omega$ . We have the diagram

$${}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}} \xleftarrow{h^\leftarrow} {}^{a\tilde{x}}\mathcal{H}_{G,\tilde{H}} \xrightarrow{h^\rightarrow} \mathrm{Bun}_{G,\tilde{H}},$$

where  $h^\rightarrow$  (resp.,  $h^\leftarrow$ ) sends the above point of  ${}^{a\tilde{x}}\mathcal{H}_{G,\tilde{H}}$  to  $(M', \mathcal{A}', V, \mathcal{C}) \in \mathrm{Bun}_{G,\tilde{H}}$  (resp., to  $(M, \mathcal{A}, V, \mathcal{C}) \in {}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}}$ ).

Restriction to  $D_x$  gives rise to the diagram

$$\begin{array}{ccccc} {}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}} & \xleftarrow{h^\leftarrow} & {}^{a\tilde{x}}\mathcal{H}_{G,\tilde{H}} & \xrightarrow{h^\rightarrow} & \mathrm{Bun}_{G,\tilde{H}} \\ \downarrow {}^a\xi & & \downarrow {}^a\xi_G & & \downarrow {}^0\xi \\ {}^a\mathcal{X}\mathcal{L} & \xleftarrow{h^\leftarrow} & {}^{a,0}\mathcal{H}_{G,\mathcal{X}\mathcal{L}} & \xrightarrow{h^\rightarrow} & {}^0\mathcal{X}\mathcal{L}, \end{array} \quad (33)$$

where the low row is the diagram (12) for  $a' = 0$ . Now Lemma 9 allows to extend (33) to the following diagram, where both squares are cartesian

$$\begin{array}{ccccc} {}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}} & \xleftarrow{h^\leftarrow} & {}^{a\tilde{x}}\mathcal{H}_{G,\tilde{H}} & \xrightarrow{h^\rightarrow} & \mathrm{Bun}_{G,\tilde{H}} \\ \downarrow {}^a\tilde{\xi} & & \downarrow {}^a\tilde{\xi}_G & & \downarrow {}^0\tilde{\xi} \\ {}^a\widetilde{\mathcal{X}\mathcal{L}} & \xleftarrow{\tilde{h}^\leftarrow} & {}^{a,0}\widetilde{\mathcal{H}}_{G,\mathcal{X}\mathcal{L}} & \xrightarrow{\tilde{h}^\rightarrow} & {}^0\widetilde{\mathcal{X}\mathcal{L}}, \end{array}$$

and the low row is the diagram (13) for  $a' = 0$  from Section 4.3.1. This provides an isomorphism

$$H_G^-(\mathcal{S}, ({}^0\tilde{\xi})^*K) \xrightarrow{\sim} ({}^a\tilde{\xi})^*H_G^-(\mathcal{S}, K)$$

functorial in  $\mathcal{S} \in {}_{-a}\mathrm{Sph}_{\mathbb{G}}$  and  $K \in D_{\mathcal{T}_0}(\tilde{\mathcal{L}}_d(W_0(F_x)))$ . Here the functors

$$({}^a\tilde{\xi})^* : D_{\mathcal{T}_a}(\tilde{\mathcal{L}}_d(W_a(F_x))) \rightarrow D({}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}})$$

are defined as in ([8], Section 7.2).

Let  ${}^{a\tilde{x}}\mathcal{H}_{\tilde{H},G}$  be the stack clasifying collections: a point of the Hecke stack  $(V, \mathcal{C}, V', \mathcal{C}', \beta) \in {}_x\mathcal{H}_{\tilde{H}}$  such that the isomorphism  $\beta$  of  $\tilde{H}$ -torsors  $(V, \mathcal{C})$  and  $(V', \mathcal{C}')$  over  $X - x$  induces an isomorphism  $\mathcal{C}(-ax) \xrightarrow{\sim} \mathcal{C}'$ ; a  $G$ -torsor  $(M, \mathcal{A})$  on  $X$  and an isomorphism  $\mathcal{A} \otimes \mathcal{C}' \xrightarrow{\sim} \Omega$ . As above, we get a diagram

$${}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}} \xleftarrow{h^{\leftarrow}} {}^{a\tilde{x}}\mathcal{H}_{\tilde{H},G} \xrightarrow{h^{\rightarrow}} \mathrm{Bun}_{G,\tilde{H}},$$

where  $h^{\rightarrow}$  (resp.,  $h^{\leftarrow}$ ) sends the above point of  ${}^{a\tilde{x}}\mathcal{H}_{\tilde{H},G}$  to  $(M, \mathcal{A}, V', \mathcal{C}')$  (resp., to  $(M, \mathcal{A}, V, \mathcal{C})$ ).

As in the case of the Hecke functor for  $G$ , we get the diagram, where both squares are cartesian

$$\begin{array}{ccccc} {}^{a\tilde{x}}\mathrm{Bun}_{G,\tilde{H}} & \xleftarrow{h^{\leftarrow}} & {}^{a\tilde{x}}\mathcal{H}_{\tilde{H},G} & \xrightarrow{h^{\rightarrow}} & \mathrm{Bun}_{G,\tilde{H}} \\ \downarrow {}^a\tilde{\xi} & & \downarrow {}^a\tilde{\xi}_{\tilde{H}} & & \downarrow {}^0\tilde{\xi} \\ {}^a\widetilde{\mathcal{X}\mathcal{L}} & \xleftarrow{\tilde{h}^{\leftarrow}} & {}^{a,0}\widetilde{\mathcal{H}_{\mathbb{H},\mathcal{X}\mathcal{L}}} & \xrightarrow{\tilde{h}^{\rightarrow}} & {}^0\widetilde{\mathcal{X}\mathcal{L}}, \end{array}$$

and the low row is the diagram (14) for  $a' = 0$  from Section 4.3.2. This provides an isomorphism

$$H_{\tilde{H}}^-(\mathcal{S}, ({}^0\tilde{\xi})^*K) \xrightarrow{\sim} ({}^a\tilde{\xi})^*H_{\tilde{H}}^-(\mathcal{S}, K)$$

functorial in  $\mathcal{S} \in {}_{-a}\mathrm{Sph}_{\mathbb{H}}$  and  $K \in D_{\mathcal{T}_0}(\tilde{\mathcal{L}}_d(W_0(F_x)))$ . By ([8], Proposition 6), we have  $({}^0\tilde{\xi})^*S_{W_0(F)} \xrightarrow{\sim} \mathrm{Aut}_{G,\tilde{H}}$  canonically. Now Theorem 2 from Theorem 3 by applying the functor  $({}^a\tilde{\xi})^*$ . Theorem 2 is proved.

5.3 In this subsection we derive Theorem 1 from Theorem 2. We give the argument only for  $m \leq n$  (the case  $m > n$  is completely similar).

Let  $a \in \mathbb{Z}$ . It suffices to establish the isomorphism (3) for any  $\mathcal{S} \in {}_{-a}\mathrm{Sph}_{\mathbb{G}}$ . By base change theorem, for  $K \in D(\mathrm{Bun}_{\tilde{H}})$  we get

$$(\pi \times \mathrm{id})^*H_G^-(\mathcal{S}, F_G(K)) \xrightarrow{\sim} ({}^a\mathfrak{p})_!({}^a\mathfrak{q}^*K \otimes H_G^-(\mathcal{S}, \mathrm{Aut}_{G,\tilde{H}}))[-\dim \mathrm{Bun}_{\tilde{H}}],$$

where  ${}^a\mathfrak{q} : {}^a\mathrm{Bun}_{G,\tilde{H}} \rightarrow \mathrm{Bun}_{\tilde{H}}$  and  ${}^a\mathfrak{p} : {}^a\mathrm{Bun}_{G,\tilde{H}} \rightarrow \tilde{X} \times \mathrm{Bun}_G$  send a collection  $(\tilde{x} \in \tilde{X}, M, \mathcal{A}, V, \mathcal{C}) \in {}^a\mathrm{Bun}_{G,\tilde{H}}$  to  $(V, \mathcal{C})$  and  $(\tilde{x}, M, \mathcal{A})$  respectively.

By Theorem 2, the latter complex identifies with

$$({}^a\mathfrak{p})_!({}^a\mathfrak{q}^*K \otimes H_{\tilde{H}}^-(\mathrm{gRes}^{\kappa}(\mathcal{S}), \mathrm{Aut}_{G,\tilde{H}}))[-\dim \mathrm{Bun}_{\tilde{H}}] \quad (34)$$

Consider the diagram

$$\begin{array}{ccccc}
\tilde{X} \times \mathrm{Bun}_{\tilde{H}} & \xleftarrow{\mathrm{supp} \times h^{\leftarrow}} & {}^a\mathcal{H}_{\tilde{H}} & \xrightarrow{h^{\rightarrow}} & \mathrm{Bun}_{\tilde{H}} \\
\uparrow \mathrm{id} \times \mathfrak{q} & & \uparrow & & \uparrow {}^a\mathfrak{q} \\
\tilde{X} \times \mathrm{Bun}_{G, \tilde{H}} & \xleftarrow{\mathrm{supp} \times h^{\leftarrow}} & {}^a_{\tilde{X}}\mathcal{H}_{\tilde{H}, G} & \xrightarrow{\mathrm{supp} \times h^{\rightarrow}} & {}^a\mathrm{Bun}_{G, \tilde{H}} \\
& \searrow \mathrm{id} \times \mathfrak{p} & & \swarrow {}^a\mathfrak{p} & \\
& & \tilde{X} \times \mathrm{Bun}_G & & 
\end{array}$$

where  ${}^a\mathcal{H}_{\tilde{H}}$  is the stack classifying  $\tilde{x} \in \tilde{X}$ ,  $\tilde{H}$ -torsors  $(V, \mathcal{C})$  and  $(V', \mathcal{C}')$  on  $X$  identified via an isomorphism  $\beta$  over  $X - \pi(\tilde{x})$  so that  $\beta$  yields  $\mathcal{C}' \xrightarrow{\sim} C(a\pi(\tilde{x}))$ . The map  $\mathrm{supp} \times h^{\leftarrow}$  (resp.,  $h^{\rightarrow}$ ) in the top row sends this point to  $(\tilde{x}, V, \mathcal{C})$  (resp., to  $(V', \mathcal{C}')$ ).

The stack  ${}^a_{\tilde{X}}\mathcal{H}_{\tilde{H}, G}$  the above diagram classifies collections:  $(\tilde{x}, V, \mathcal{C}, V', \mathcal{C}', \beta) \in {}^a\mathcal{H}_{\tilde{H}}$ , a  $G$ -torsor  $(M, \mathcal{A})$  on  $X$ , and an isomorphism  $\mathcal{A} \otimes \mathcal{C} \xrightarrow{\sim} \Omega$ . The map  $\mathrm{supp} \times h^{\leftarrow}$  (resp.,  $\mathrm{supp} \times h^{\rightarrow}$ ) in the middle row sends this collection to  $(\tilde{x}, M, \mathcal{A}, V, \mathcal{C})$  (resp., to  $(\tilde{x}, M, \mathcal{A}, V', \mathcal{C}')$ ).

By the projection formulas, now (34) identifies with

$$(\mathrm{id} \times \mathfrak{p})_!(\mathrm{Aut}_{G, \tilde{H}} \otimes (\mathrm{id} \times \mathfrak{q})^* \mathrm{H}_{\tilde{H}}^{\leftarrow}(\mathrm{gRes}^{\kappa}(\mathcal{S}), K))[-\dim \mathrm{Bun}_{\tilde{H}}]$$

Theorem 1 is proved.

## APPENDIX A. INVARIANTS IN THE CLASSICAL SETTING

A.1 In this appendix we assume that  $k_0 \subset k$  is a finite subfield, and all the objects introduced in Section 4 are defined over  $k_0$ . Write  $F_0$  for  $k_0$ -valued points of  $F$ . Our purpose is to prove Proposition A.1 formulated in Section 4.8.7.2.

**Lemma 10.** *Let  $G$  be a group scheme over  $\mathrm{Spec} \mathcal{O}$ ,  $P \subset G$  be a parabolic and  $U \subset P$  its unipotent radical. Let  $V$  be a smooth  $\mathbb{Q}_{\ell}$ -representation of  $G(F)$ . Then the natural map  $V^{G(\mathcal{O})} \rightarrow V_{U(F)}$  is injective, here  $V_{U(F)}$  denotes the corresponding Jacquet module.*

*Proof* The author thanks J.-F. Dat for the following proof communicated to me. Pick a Borel subgroup  $B \subset P$ , write  $I \subset G(\mathcal{O})$  for the corresponding Iwahori subgroup. It suffices to show that  $V^I \rightarrow V_{U(F)}$  is injective.

Let  $v \in V^I$  vanish in  $V_{U(F)}$ . Then one may find a semisimple  $t \in B(F)$  such that the characteristic function  $\phi$  of  $ItI$  annihilates  $v$  (it suffices that the action of  $t$  on  $U(F)$  be sufficiently contracting). However,  $\phi$  is invertible in the Iwahori-Hecke algebra of  $(G(F), I)$ , so  $v = 0$ .  $\square$

**Lemma 11.** *The maps  $J_{P_{\mathbb{H}_a}}^* : \mathrm{Weil}_a(k_0) \rightarrow \mathcal{S}_{Q\mathbb{H}_a(\mathcal{O})}(Q\Upsilon_a(F_0))$  and*

$$J_{P_{G_a}}^* : \mathrm{Weil}_a(k_0) \rightarrow \mathcal{S}_{Q\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F_0)) \quad (35)$$

*are injective.*

*Proof* Both claims being similar, we prove only the second one. Apply Lemma 10 for the parabolic  $P_{\mathbb{H}_a} \subset \mathbb{H}_a$  and the representation  $\mathcal{S}(\Pi_a(F))$  of  $\mathcal{T}_a(F)$ . Remind that  $\mathcal{T}_a = \{(g_1, g_2) \in \mathbb{G}_a \times \mathbb{H}_a \mid (g_1, g_2) \text{ acts trivially on } A_a \otimes C_a\}$ , and  $U_{\mathbb{H}_a} \subset P_{\mathbb{H}_a}$  is the unipotent radical.

For  $v \in \Pi_a(F)$  let  $s_{\Pi}(v) : C_a^* \otimes \wedge^2 U_a(F) \rightarrow \Omega(F)$  be the map introduced in Section 4.8.4. Write  $\text{Cr}(\Pi_a)$  for the space of  $v \in \Pi_a(F)$  such that  $s_{\Pi}(v) = 0$ . By ([10], page 72), the Jacquet module  $\mathcal{S}(\Pi_a(F))_{U_{\mathbb{H}_a}(F)}$  identifies with the Schwarz space  $\mathcal{S}(\text{Cr}(\Pi_a))$ , and the projection

$$\mathcal{S}(\Pi_a(F)) \rightarrow \mathcal{S}(\Pi_a(F))_{U_{\mathbb{H}_a}(F)}$$

identifies with the restriction map  $\mathcal{S}(\Pi_a(F)) \rightarrow \mathcal{S}(\text{Cr}(\Pi_a))$ . We learn that the restriction map  $\text{Weil}_a(k_0) \rightarrow \mathcal{S}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\text{Cr}(\Pi_a))$  is injective. So, (35) is also injective.  $\square$

*Proof of Proposition A.1*

For  $b \in \mathbb{Z}$  set  ${}_b\mathcal{H}_{\mathbb{G}} = K_0({}_b\text{Sph}_{\mathbb{G}}) \otimes \bar{\mathbb{Q}}_{\ell}$  and  ${}_b\mathcal{H}_{Q(\mathbb{G})} = K_0({}_b\text{Sph}_{Q(\mathbb{G})}) \otimes \bar{\mathbb{Q}}_{\ell}$ . So,

$$\mathcal{H}_{\mathbb{G}} = \bigoplus_{b \in \mathbb{Z}} {}_b\mathcal{H}_{\mathbb{G}}, \quad \mathcal{H}_{Q(\mathbb{G})} = \bigoplus_{b \in \mathbb{Z}} {}_b\mathcal{H}_{Q(\mathbb{G})}$$

are the Hecke algebras for  $\mathbb{G}$  and  $Q(\mathbb{G})$  respectively. From Proposition 4, we learn that the map

$$-{}_a\mathcal{H}_{Q(\mathbb{G})} \rightarrow \mathcal{S}_{Q\mathbb{G}\mathbb{H}_a(\mathcal{O})}(Q\Pi_a(F_0))$$

given by  $\mathcal{S} \mapsto \text{tr}_{k_0} H_{Q(\mathbb{G})}^{\leftarrow}(\mathcal{S}, I_0)$  is an isomorphism of  $\bar{\mathbb{Q}}_{\ell}$ -vector spaces. Write  $-{}_aW \subset -{}_a\mathcal{H}_{Q(\mathbb{G})}$  for the image of the map (35). We get a  $\mathbb{Z}$ -graded subspace  $W := \bigoplus_{a \in \mathbb{Z}} {}_aW \subset \mathcal{H}_{Q(\mathbb{G})}$ .

For  $a, a' \in \mathbb{Z}$  we have the Hecke operators

$$H_{\mathbb{G}}^{\leftarrow} : {}_{a'-a}\mathcal{H}_{\mathbb{G}} \times \mathcal{S}_{\mathbb{G}Q\mathbb{H}_{a'}(\mathcal{O})}(\Pi_{a'}(F_0)) \rightarrow \mathcal{S}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F_0))$$

defined as in Section 4.8.1. We claim that for  $\mathcal{S} \in {}_{a'-a}\mathcal{H}_{\mathbb{G}}$  the operator  $H_{\mathbb{G}}^{\leftarrow}(\mathcal{S}, \cdot)$  sends  $\text{Weil}_{a'}(k_0)$  to the subspace  $\text{Weil}_a(k_0) \subset \mathcal{S}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F_0))$ . This follows from the fact the actions of the groupoids  $\mathbb{G}Q\mathbb{H}$  and  $\mathbb{H}Q\mathbb{G}$  on the spaces  $\mathcal{S}_{\mathbb{G}Q\mathbb{H}_a(\mathcal{O})}(\Pi_a(F_0))$  commute with each other.

More precisely, for  $a, b \in \mathbb{Z}$  given  $g = (g_1, g_2) \in \mathcal{T}_{b,a}$  such that  $g_2 : V_a \xrightarrow{\sim} V_b$  is an isomorphism of  $Q(\mathbb{H})$ -torsors over  $\text{Spec } \mathcal{O}$ , let  $h = (h_1, h_2) \in \mathcal{T}_b$  be any element such that  $h_1 : M_b \xrightarrow{\sim} M_b$  is a scalar automorphism of the  $\mathbb{G}$ -torsor  $M_b$  over  $\text{Spec } \mathcal{O}$ . Here  $h_2$  is an automorphism of the  $\mathbb{H}$ -torsor  $V_b$  over  $\text{Spec } \mathcal{O}$ . Set  $h'_2 = g_2^{-1}h_2g_2$ , so  $h'_2$  is an automorphism of the  $\mathbb{H}$ -torsor  $V_a$  over  $\text{Spec } \mathcal{O}$ . Set  $h'_1 = h_1$  then  $h' = (h'_1, h'_2) \in \mathcal{T}_a$ . The equality  $gh' = hg$  in  $\mathcal{T}$  shows that  $g : \mathcal{S}(\Pi_a(F)) \rightarrow \mathcal{S}(\Pi_b(F))$  sends  $\mathbb{H}_a(\mathcal{O})$ -equivariant objects to  $\mathbb{H}_b(\mathcal{O})$ -equivariant objects. We have used the action of the groupoid  $\mathcal{T}$  on the spaces  $\mathcal{S}(\Pi_a(F))$  obtained as in Remark 5.

Thus,  $W$  is a  $\mathbb{Z}$ -graded module over the  $\mathbb{Z}$ -graded ring  $\mathcal{H}_{\mathbb{G}}$ . We also know from ([7], Proposition 2) that  ${}_0W = {}_0\mathcal{H}_{\mathbb{G}}$ . Our statement is reduced to Lemma 12 below.  $\square$

Remind that we have picked a maximal torus  $T_{\mathbb{G}} \subset Q(\mathbb{G})$ . Write  $W$  (resp.,  $W_Q$ ) for the Weyl group of  $(\mathbb{G}, T_{\mathbb{G}})$  (resp., of  $(Q(\mathbb{G}), T_{\mathbb{G}})$ ). Then

$$\mathcal{H}_{Q(\mathbb{G})} \xrightarrow{\sim} \bar{\mathbb{Q}}_{\ell}[\check{T}_{\mathbb{G}}]^{W_Q}, \quad \mathcal{H}_{\mathbb{G}} \xrightarrow{\sim} \bar{\mathbb{Q}}_{\ell}[\check{T}_{\mathbb{G}}]^W$$

The homomorphism  $\text{Res}^{\kappa_{\mathbb{G}}} : \mathcal{H}_{\mathbb{G}} \rightarrow \mathcal{H}_{Q(\mathbb{G})}$  (cf. Section 4.8.6) comes from the map  $f^{\kappa_{\mathbb{G}}} : \check{T}_{\mathbb{G}}^{W_Q} \rightarrow \check{T}_{\mathbb{G}}^W$  obtained by taking the Weil group invariants of the map  $\check{T}_{\mathbb{G}} \rightarrow \check{T}_{\mathbb{G}}$ ,  $t \mapsto t\nu(q^{1/2})$ , where  $\nu$  is some coweight of the center  $Z(\check{Q}(\mathbb{G}))$ , and  $q$  is the number of elements of  $k_0$ .

**Lemma 12.** *View  $\mathcal{H}_{Q(\mathbb{G})}$  as a  $\mathbb{Z}$ -graded  $\mathcal{H}(\mathbb{G})$ -module via  $\text{Res}^{\kappa_{\mathbb{G}}} : \mathcal{H}_{\mathbb{G}} \rightarrow \mathcal{H}_{Q(\mathbb{G})}$ . Let  $W = \bigoplus_{a \in \mathbb{Z}} {}_a W \subset \mathcal{H}_{Q(\mathbb{G})} = \bigoplus_{a \in \mathbb{Z}} {}_a \mathcal{H}_{Q(\mathbb{G})}$  be a  $\mathbb{Z}$ -graded submodule over the  $\mathbb{Z}$ -graded ring  $\mathcal{H}_{\mathbb{G}}$ . Assume that  ${}_0 W = {}_0 \mathcal{H}_{\mathbb{G}}$ . Then  $W = \mathcal{H}_{\mathbb{G}}$ .*

*Proof* Given  $x \in {}_a W$ , pick a nonzero  $h \in -{}_a \mathcal{H}_{\mathbb{G}}$  then  $hx \in {}_0 \mathcal{H}_{\mathbb{G}}$ . So,  $x$  is a rational function on  $\check{T}_{\mathbb{G}}^W$  which becomes everywhere regular after restriction under  $f^{\kappa_{\mathbb{G}}} : \check{T}_{\mathbb{G}}^{W_Q} \rightarrow \check{T}_{\mathbb{G}}^W$ . Since  $\check{T}_{\mathbb{G}}^W$  is normal by Remark 8 below, and  $x$  is entire over  $\mathbb{Q}_{\ell}[\check{T}_{\mathbb{G}}]^W$ , it follows that  $x \in \mathbb{Q}_{\ell}[\check{T}_{\mathbb{G}}]^W$ .  $\square$

*Remark 8.* Let  $A$  be an entire normal ring,  $W$  be a finite group acting on  $A$ . Assuming that  $A$  is finite over  $A^W$ , one checks that  $A^W$  is normal.

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